Lectures 11, 12, 13, 14

Electrostatic fields in matter, conductors

a) Electrons in metals form the so-called "electron liquid," which is able to flow under an arbitrarily small electric field. (That is, any \( \vec{E} \) install a conductor causes an electric current.) Therefore there cannot be an electric field inside a conductor in a stationary situation:

\[ t = 0: \text{Conductor is placed in } \vec{E} \text{ field} \]

\[ \text{current starts to flow} \]

\[ t > 0: \]

\[ \vec{E}_{\text{induced}} \]

\[ \vec{E}_{\text{total}} = \vec{E}_{\text{external}} + \vec{E}_{\text{induced}} \]

| \( |\vec{E}_{\text{total}}| < |\vec{E}_{\text{external}}| \) |

\[ t \rightarrow \infty: \text{(really a few picoseconds at most)} \]

\[ \text{no field inside, current flow has stopped} \]

\[ \vec{E}_{\text{inside}} = \vec{E}_{\text{external}} + \vec{E}_{\text{induced}} = 0 \]
From these qualitative considerations we can derive a few conclusions:

1) \( \vec{E}_{\text{inside}} = 0 \) for a conductor.

2) (1) implies that the surface of a conductor is an equipotential one:

Physically, it is the case that the charge flows around the surface.

Formally, \( \Phi_b - \Phi_a = -\int_{\Gamma_a} \vec{E} \cdot d\vec{r} = 0 \), for any two points, since all potentials are the same.

Inside the conductor the potential is exactly as on its surface by the same arguments.

3) \( P_{\text{inside}} = 0 \) : \( \vec{D} \cdot \vec{E} = \frac{\varepsilon}{\varepsilon_0} P \Rightarrow P_{\text{inside}} = \boxed{0} \) - no charge inside the conductor, that is, the charge of ions is exactly balanced by the charge of the electrons.

4) If we place a large external charge into a bulk conductor it ends up on its surface to minimize its electrostatic energy.
5) Field lines outside the conductor are 1 to its surface.

Physically, if there were a component of the electric field along the surface (tangential component), it would make the charges that lay along the surface, which cannot happen on a static surface.

6) Electrostatic induction uniqueness theorem

the redistribution of charges on an object caused by an electric field is called the electrostatic induction; the charges are said to be induced by the field.

Uniqueness theorem states that if we found one solution of an electrostatic problem, (that is, of the Poisson equation \(-\nabla^2 \phi = \rho / \varepsilon_0\) supplemented with the appropriate boundary conditions) it is guaranteed to be the only solution.

Corollary: if we guessed a solution and it works, we can be sure we have guessed the only solution.
Let us consider a few illustrative situations.

1) charged object + neutral body
charges of the same sign as that of A tend to get farther from it, the opposite ones are drawn closer.

Two consequences:
1) since there is no field inside, there are no field lines, "canceling" +/− induced charges. Thus it looks like the conductor "cut" some of the field lines.
2) since the charges of the opposite sign are on average closer to the charged object than the same-sign ones, the charged object is attracted to a neutral one. This does not depend on the sign of the charge of that object.

(This is a very important fact)
ii) Empty cavity. Shielding.

Consider a conductor with a cavity placed in an external field. Are there any charges on the surface of a cavity? Is there any field in it? The answers to both questions are no.

Let us assume there are charges on a cavity's surface (there still cannot be any in the bulk of the conductor). Then field lines have to start/end at them. But this leads to a contradiction, as then we can make closed paths that go through the bulk of the conductor (where \( \mathbf{E} = 0 \), and the line integral of \( \mathbf{E} \) is not zero along such closed paths, hence there is neither charges nor fields. The cavity is completely screened from the external field. If one wants to shield a sensitive device from external fields, one way to accomplish this would be to surround that device with a conductive shell.
Method of images

This method will allow us to find electric fields created by charged objects in the presence of conductors under certain circumstances.

Let us start with a specific example.

a) Charge near a conductor in halfspace

The presence of +q will cause some charge redistribution on the surface - induce some charge density - and thus the picture of the field lines will be altered. Our task is to find what the E-field looks like outside the conductor.

Indeed, we know what field looks like inside: it is simply zero. Also we know that the surface of the conductor is an equipotential one. Let us sketch the field lines to get an idea of what is going on:
we note that the field lines look a lot like those for a pair of charges \( \pm q \), the \(-q\) one being placed below the surface, at the same distance from it as \(+q\).

Thus our guess: The field of the system outside the conductor looks like the one of the original charge, \(+q\), and its mirror image: charge \(-q\) placed at same distance below the surface of the conductor.

NB: always remember that this construction works only outside the conductor. Inside the field is zero, unlike that of the two point charges.

Let's check if our guess solves the problem.

- Obviously, the field outside satisfies \( \mathbf{E} = \frac{1}{2} q \mathbf{E}_0 = \frac{1}{2} q \mathbf{E}_0 \), as \(-q\) is below the surface.

- We need to check that the field satisfies the boundary condition \( \psi (z = 0) = \text{const.} \)

\[
\psi (\mathbf{r}) = \frac{1}{4 \pi \varepsilon_0} \cdot \frac{q}{r} + \frac{1}{4 \pi \varepsilon_0} \cdot \frac{-q}{r} = \psi \left( \frac{+q}{-q} \right)
\]

This works!

By uniqueness theory, there is no other solution! Again, this theorem is as awesome as they say.
NB: This method has its limitations. Even if we found a solution in terms of mirror charges, it would be wrong to say, calculate the energy using the mirror charge distribution.

Indeed, \[ W = \frac{E_0}{\pi} \int d\mathbf{r} \mathbf{E}^2 \]. For a system of two charges, \( \mathbf{E} \) fills up the entire space, but in reality, it is non-zero only where there is no conductor, thus is twice as small as for a system of two point charges, of the opposite signs, placed at the distance of 2d from each other.

Another example: point charge near a conducting sphere.

Consider a point charge placed near a grounded conducting sphere. What is the system of charges that would represent the electric field outside the sphere (it is zero inside, again).

Let us make a guess that the image charge is located on the line connecting \( q \) to the center of the sphere, \( r \) away from the latter.
we determine the magnitude of \( q' \) by demanding that the sphere is an equipotential surface. The potential on it is equal to zero if infinity is chosen as the reference point.

For a point on the sphere at angle \( \theta \) to \( q' \), the potential is

\[
\Phi(\theta) = 0 = \frac{1}{4\pi \varepsilon_0} \left[ \frac{q}{\sqrt{(r - r')^2 + r^2 + 2rr' \cos \theta}} + \frac{q'}{\sqrt{r'^2 + r^2 + 2rr' \cos \theta}} \right]
\]

There seemingly a problem with this equation: there are two unknowns, \( q \) and \( r' \), and only one equation. The problem is resolved by observing that \( \Phi(\theta) = 0 \) must hold for any \( \theta \). This will effectively make it two equations: one for terms that do not depend on \( \cos \theta \), and one for those that do:

\[
\frac{q}{\sqrt{r^2 - r'^2}} + \frac{q'}{\sqrt{r'^2 + r^2}} = 0 \Rightarrow \frac{r^2 + r'^2 + 2rr' \cos \theta}{r^2 + r'^2 + 2rr' \cos \theta} = \left( \frac{q'}{q} \right)^2, \quad q, q' < 0
\]

and we get that

\[
r^2 + r'^2 + 2rr' \cos \theta = \left( \frac{q'}{q} \right)^2 \left( r^2 + r'^2 + 2rr' \cos \theta \right)
\]

Collecting terms that do not contain \( \cos \theta \), and those that do, we obtain
\[ r'^2 + r^2 = \left(\frac{q'}{q}\right)^2 \left( r^2 + r'^2 \right) \]

\[ 2r'r = \left(\frac{q'}{q}\right)^2 2rr' \]

From here we get

\[ \frac{r'^2 + r^2}{r'} = \frac{r^2 + r'^2}{r} \Rightarrow \left(\frac{q'}{q}\right)^2 = \frac{r'}{r}. \]

\[ r^2 + r'^2 - \frac{r^2 + r'^2}{r^2} r' = 0. \quad r' = \frac{r^2}{r'} \pm \sqrt{\left(\frac{r+R^2}{r'}\right)^2 - 4R^2} \]

\[ = \frac{1}{2} \left( \frac{r^2}{r'} + \left(\frac{r-R^2}{r'}\right) \right). \]

We see that there are two solutions:

1) \( r' = r, \quad q' = -q; \quad (q/|q| < 0!) \)

2) \( r' = \frac{R^2}{r}, \quad q' = -q, \frac{R}{r}; \)

The first one is unphysical, as it corresponds to an image charge of \(-q\) sitting right on top of \(q\), killing E-field everywhere, which is obviously not the case. The solution is

\[ r' = \frac{R^2}{r}, \quad q' = -q, \frac{R}{r}. \]

This kind of image is called inversion in the sphere (\(rr'=r^2\)).
Surface charge density and forces acting on conductors.

If we know the solution to the electrostatic problem in the presence of conducting bodies, we can calculate the surface charges induced on them. Consider a small "pillbox" Gaussian surface around a small patch of the surface, such that the top and bottom sides of the pillbox are parallel to the patch.

By making the height of the pillbox $\rightarrow 0$, we note there is no flux through the side surfaces (this holds for any surface). Thus by Gauss’s law we get

$$\left( E_{n,\text{outside}} - E_{n,\text{inside}} \right) \cdot \Delta S = \frac{1}{\varepsilon_0} \cdot \sigma \Delta S \quad \text{or}$$

$$\sigma = \varepsilon_0 \left( E_{n,\text{outside}} - E_{n,\text{inside}} \right) \quad \text{The field is discontinuous if there is surface charge.}$$

This is a general expression, but for metals $E_{n,\text{outside}} = 0$, so we get

$$\sigma = \varepsilon_0 \cdot E_{n,\text{outside}} = -\varepsilon_0 \frac{\partial \Phi}{\partial n} \quad \text{(where } E = -\varepsilon_\Phi)$$

$\frac{\partial \Phi}{\partial n}$ denotes the projected derivative of $\Phi$ in the direction $\perp$ to the surface. e.g. $\frac{1}{x} \overset{\partial \Phi}{\partial x} \overset{\partial \Phi}{\partial z} \overset{\partial \Phi}{\partial \gamma} = \frac{\partial \Phi}{\partial n}$
**Force acting on a conductor**

The knowledge of \( \sigma \) allows us to find the force acting on an element of the surface. We would naively try to multiply the surface charge by the electric field at the surface. But the latter is discontinuous! Which field do we take?

To resolve the difficulty, we note that the charge cannot exert force on itself, thus in any case we need to single out the field extrinsic to the surface charge, that is, not created by it. In fact, the discontinuity is entirely due to the surface charge, see the fig. from the previous page. Thus when its field removed, the discontinuity is gone:

![Diagram showing the relationship between the electric field and the surface charge](image)

Consider a conductor in an external field. We know that the electric field points normal to the surface:

\[
\vec{E}_{\text{total}} = \vec{E}_{\text{ext}} + \frac{1}{2 \varepsilon_0} \sigma \vec{n}.
\]

Then we have outside:

\[
\vec{E}_{\text{ext}} = \vec{E}_{\text{ext}} + \frac{1}{2 \varepsilon_0} \sigma \vec{n}
\]

and inside:

\[
0 = \vec{E}_{\text{ext}} - \frac{1}{2 \varepsilon_0} \sigma \vec{n}
\]

\[
\Rightarrow \vec{E}_{\text{ext}} = \frac{\vec{E}_{\text{total}}}{\sigma}
\]
Thus the force acting on a surface element is

\[ \Delta \mathbf{F} = \frac{1}{2} \mathbf{E}_{\text{total}} \cdot \sigma \Delta S = \frac{1}{2} (\mathbf{E}_{\text{out}} + \mathbf{E}_{\text{in}}) \cdot \sigma \Delta S \text{ in general} \]

we can further rewrite this as an expression for a larger unit area.

\[ \mathbf{F} = \frac{\Delta \mathbf{F}}{\Delta S} = \frac{1}{2} \nabla \mathbf{E}_{\text{total}} \cdot \sigma = \frac{1}{2 \varepsilon_0} \sigma \mathbf{n} \]

Note that the force disappears does not depend on the sign of \( \sigma \): a conductor is always drawn into electric field.

**Example:** Charge near a conducting half-space: we have solved this problem by the method of images, thus

\[ \Phi \text{ point-sol} = \Phi_0 = \frac{1}{4 \pi \varepsilon_0} \cdot \frac{q}{\sqrt{x^2+y^2+(z-d)^2}} \]

\[ -\frac{1}{4 \pi \varepsilon_0} \cdot \frac{9}{\sqrt{x^2+y^2+(z+t-\bar{d})^2}} \]

Surface charge density:

in this case, \( \mathbf{n} = \hat{z} \), thus

\[ \sigma = -\frac{x}{4 \pi \varepsilon_0} \cdot \frac{\partial \Phi_0}{\partial z} \bigg|_{z=\infty} = -\frac{9 \varepsilon_0}{4 \pi} \cdot \left( \frac{(z-d)}{\sqrt{r^2+(z-d)^2}} - \frac{(z+t)}{\sqrt{r^2+(z+t-d)^2}} \right) \]

\[ = -\frac{9}{4 \pi} \cdot \frac{d}{(r^2+d^2)^{3/2}} \]

where \( r^2 = x^2 + y^2 \).
we see that \( \sigma(r) \propto \frac{1}{\sqrt{r}} \) as \( r \to \infty \), fast decay.

Also note the negative sign of \( \sigma \), as expected for \( +q > 0 \).

**Total induced charge**:

\[
\Phi = \int_0^\infty d\ell \int_0^\infty d\theta \cdot \frac{-q}{2\pi r \ell} = -q \int_0^\infty dx \cdot x \cdot \frac{1}{(1+x^2)^{3/2}} = -q
\]

Again, as expected.

**Total force acting on the surface**:

\[
F = \frac{1}{2} \sigma \hat{E} \cdot \hat{N}
\]

Thus,

\[
\hat{F} \sigma = \frac{1}{2} \hat{N} \cdot \int d\ell d\theta \cdot \frac{q^2}{4\pi^2} \cdot \frac{d^2}{(r^2 + d^2)^3} = \frac{1}{2} \hat{N} \cdot \frac{q^2}{4\pi \varepsilon_0} \cdot \frac{1}{(2d)^2}
\]

This is exactly what we expect: the image charge attracts the original one with the force

\[
\hat{F}_q = -\frac{1}{2} \cdot \frac{q^2}{4\pi \varepsilon_0 (2d)^2}, \text{ thus the original force}
\]

one must be pulling the entire metal up with the same force, as obtained.