Electrostatics of dielectrics

What happens when a non-conducting body is brought into an electric field? To see what happens, let's look inside it.

In a dielectric material, electrons can be thought of as tightly bound to ionic cores, so all they can do is to get displaced a bit when put into an electric field (as opposed to the case of metals where they flow under arbitrarily small fields).

Before:

\[ \text{ induced dipole} \]

After:

We say that dielectrics/molecules in them get polarized in electric fields.

The behaviour just outlined is that of non-polar molecules, but there are examples of molecules with built-in dipole moment - polar ones, see below.

We will view a dielectric/insulator as a collection of such polar or non-polar molecules (or atoms).
2. **Molecules and atoms in electric fields**

a) **General considerations:**

Consider a molecule as a collection of positive and negative charges: \( \{ q_i^+, r_i^+ \} \), \( \{ q_i^-, r_i^- \} \).

In the case of electrons, \( r_i^- \) should be understood as "time averaged" positions, as electrons undergo rapid motion inside atoms.

We can define the "center of charge" positions analogously to the "center of mass" in a system of point masses:

\[
\mathbf{r}^+ = \frac{\sum_i q_i^+ \mathbf{r}_i^+}{\sum_i q_i^+}, \quad \mathbf{r}^- = \frac{\sum_i q_i^- \mathbf{r}_i^-}{\sum_i q_i^-}
\]

In the absence of an electric field, \( \mathbf{r}^+ = \mathbf{r}^- \) can either coincide or not.

**Examples:**

\( \text{H}_2\text{O} \):

\[
\begin{align*}
\text{H}_2\text{O} & : \quad \mathbf{r}^+ \neq \mathbf{r}^- - \text{polar molecule} \\
\end{align*}
\]
Thus a polar molecule has a build-in dipole moment:
\[
\vec{d} = \vec{r}^+ - \vec{r}^-, \quad \vec{p} = \vec{d} \cdot q = \sum_i \vec{r}^+_i q^+_i + \sum_j \vec{r}^-_j q^-_j
\]

\[ q = \sum_i q^+_i \]

**NB:** For a neutral object, \( \sum_i q^+_i + \sum_j q^-_j = 0 \), the dipole moment does not depend on the choice of the origin:

\[
\vec{r}^+ \rightarrow \vec{r}^+ + \vec{a} \quad \text{shift of the origin},
\]

then
\[
\vec{p}' = \sum_i \vec{r}^+_i q^+_i + \sum_j \vec{r}^-_j q^-_j = \vec{p} - \sum_i (\vec{r}^+_i + \vec{r}^-_i) q^+_i + \sum_j (\vec{r}^-_j + \vec{r}^-_j) q^-_j = \vec{p} + \vec{a} \left( \sum_i q^+_i + \sum_j q^-_j \right) = \vec{p} + \vec{a} \cdot 0 = \vec{p}
\]

To summarize, we introduce a canonical index for all charges to make
\[
\vec{p} = \sum_k q_k \vec{r}_k
\]

b) **Non-polar molecule in an electric field.**

- A dipole moment appears usually \( \vec{p} \propto \vec{E} \), but there can be non-linear corrections.
One can write \( \vec{p} = \vec{d} \cdot \vec{E} \) where \( \vec{d} \) is a coeft, called the polarizability or a molecular.

What is \([\vec{d}]\)?

\[
[p] = 0.1 \text{ Nm}, \quad [\vec{E} \cdot \vec{d}] = \frac{0}{L^2} \implies [\vec{d}] = L^2
\]

Polar molecules in an electric field:
The built-in dipole moment is usually only very nearly affected by an external electric field (it is said to be rigid). Thus the only effect would be a re-orientation of the dipole, see below.

2) Dipole in an electric field: torques and forces.

Consider a uniform field first.

The forces acting on the dipole are shown in the Fig.

In a uniform field,
\( |F_+| = |F_-| \), since
\( \vec{F}_+ = \pm q\vec{E} \), and \( \vec{E} \) is the same for the two charges. Thus there is not net force but only a torque acting on the dipole trying to align it with the electric field:

\[
\vec{M} = \sum_i \vec{r}_i \times \vec{F}_i = q \vec{F}_+ \times \vec{E} - q \vec{F}_- \times \vec{E} = \vec{p} \times \vec{E}
\]
If the electric field is non-uniform, the forces are no longer equal, and there is a net force on a dipole. Qualitatively, it is clear that the force is larger for a charge that is in a higher E-field, and that force will dominate the behavior of the dipole.

If the angle \( \theta \) between \( \vec{D} \) and \( \vec{E} \) is acute, the dipole is "sucked into the field" (case a). If \( \theta \) is obtuse, the dipole is pushed out of it.

However, if the dipole is free to rotate, the torque will align it with the electric field, and such a dipole will always be sucked into the region of a stronger field.

More quantitatively,

\[
\vec{F}_{net} = \vec{F}_+ + \vec{F}_- = q \vec{E} (\vec{r} + \frac{\vec{D}}{2}) - q \vec{E} \left( \vec{r} - \frac{\vec{D}}{2} \right) = q \left( \frac{\vec{D}}{2} \cdot \vec{E} \right) - q \left( - \frac{\vec{D}}{2} \cdot \vec{E} \right) = (\vec{D} \cdot \vec{E}) \vec{E}
\]

\[
\vec{D} = \vec{p}_x \hat{x} + \vec{p}_y \hat{y} + \vec{p}_z \hat{z} \quad (\vec{p}_x = q_x, \text{ etc})
\]

\[
\vec{F}_{net} = (\vec{D} \cdot \vec{E}) \vec{E}
\]