Lecture 20

Energy in dielectric system

When we talked about energy of electric field in vacuum, we calculated it as work done while assembling the charge distribution that creates that field. When dielectric medium is present we have to count not only the energy of the field itself, but also the "mechanical" energy needed to "stretch" the molecules of the dielectric in the process of polarization. That is we still bring in "free charges", but have to calculate both the energy of the resultant $E$-field and the energy of polarized dielectric, which is not related to the electric field.

We will consider a particular model of a dielectric medium, but the result will be correct in the general case.

In this model, we view molecules of the dielectric as $\pm$-charge connected with a spring of rigidity $K$, which is stretched by the electric field:

\[ \Delta x \approx qE \Rightarrow \Delta x = \frac{qE}{K} \Rightarrow \beta = q\Delta x = \frac{qE}{K} \]

The resultant polarization is

\[ \mathbf{P} = n\beta = \frac{nq^2}{K} \mathbf{E} \equiv \varepsilon_0 \varepsilon \mathbf{E} \], or $\varepsilon = \frac{1}{\varepsilon_0} \cdot \frac{nq^2}{K}$ model.
The total energy of the system is then

\[ W = \oint \text{Work} + W_{\text{mechanical}} = \]

\[ = \int \text{div} \cdot \frac{\varepsilon_0 \mathbf{E}^2}{2} + \int \text{grad} \cdot \nabla \cdot \frac{K \Delta x^2}{2} = \int \text{div} \cdot \frac{\varepsilon_0 \mathbf{E}^2}{2} + \]

\[ + \int \text{div} \cdot \nabla \cdot \frac{K}{\varepsilon_0 K^2} \mathbf{E}^2 = \int \text{div} \cdot \frac{\varepsilon_0 \mathbf{E}^2}{2} + \int \text{div} \cdot \frac{\varepsilon_0 \mathbf{E}^2}{2} = \]

\[ = \int \text{div} \cdot \frac{\varepsilon_0 \mathbf{E}^2}{2} = \left[ \int \text{div} \cdot \frac{\varepsilon \mathbf{E}^2}{2} \right]. \]

This result is more general than the model it is obtained from.

We can make one further step by requiring that \(\mathbf{E} = \mathbf{E}(\mathbf{r})\) is the displacement inside a linear dielectric:

\[ W = \int \text{div} \cdot \frac{1}{2} \varepsilon \mathbf{E}^2 \] this formula is valid in the most general case, regardless of the nature of the dielectric.

Example: energy stored in a parallel-plate capacitor filled with a dielectric.

\[ W = \int \text{div} \cdot \frac{\varepsilon_0 \mathbf{E}^2}{2} = \int \text{div} \cdot \frac{\varepsilon_0}{2} \left( \frac{\partial \mathbf{E}}{\partial t} \right)^2 = \]

\[ = \frac{\varepsilon_0}{2} A \cdot d \cdot \frac{\Delta \mathbf{E}^2}{d^2} = \frac{1}{2} \frac{\varepsilon_0}{d} \Delta \mathbf{E}^2 = \frac{1}{2} C \Delta \mathbf{E}^2, \]
where \( C = \varepsilon \frac{A}{d} \) is the capacitance.

If we want to express \( W \) through the charge on the plates, we get:

\[
W = A \cdot d \cdot \frac{\varepsilon}{2} \left( \frac{1}{\varepsilon} \cdot \frac{Q^2}{A^2} \right)^2 = \frac{1}{2} \frac{Q^2}{C}.
\]

(Electric field inside the capacitor)

We see that we still have the expressions from the vacuum case,

\[
W = \frac{1}{2} C Q^2 = \frac{1}{2} \varepsilon \frac{Q^2}{C}, \quad \text{but} \quad C = \varepsilon \frac{A}{d} \quad \text{instead of} \quad \varepsilon \varepsilon_0 \frac{A}{d} \quad \text{in the vacuum case}.
\]