The Periodic Standing Wave Approximation: Adapted coordinates and spectral methods

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Abstract

The periodic standing wave method for the binary inspiral of black holes and neutron stars computes exact numerical solutions for periodic standing wave spacetimes and then extracts approximate solutions of the physical problem, with outgoing waves. The method requires solution of a boundary value problem with a mixed (hyperbolic and elliptic) character. We present here an approach based on a coordinate system adapted to the geometry of the problem. The new coordinates are most promising in connection with a spectral method. Results presented for nonlinear model problems suggest that the advantages of the adapted coordinates and spectral method might warrant the extra complexity they entail.

I. INTRODUCTION

A. Background

The detection and interpretation of gravitational wave signals, from inspiralling black holes or neutron stars, requires a solution of Einstein’s equations for the late stages of the inspiral [1]. Much effort is going into the development of computer codes that will evolve solutions forward in time. Such codes will eventually provide the needed answers about the strong field interaction and merger of the binary objects, but many technical challenges of such a computation slow the development of the needed codes. This has led us to propose as a near term alternative, the periodic standing wave (PSW) approach. Elements of this approximation have been introduced elsewhere [2, 3], but are most thoroughly presented in a recent paper [4] that we will hereafter refer to as “Paper I.” In the PSW approach, a numerical solution is sought to Einstein’s equations, not for a spacetime geometry evolved from initial data, but rather for sources and fields that rotate rigidly (i.e., with a helical Killing vector) and that are coupled to standing waves.

Paper I gives the details of how to extract from this solution an approximation to the problem of interest: a slowly inspiralling pair of objects coupled to outgoing waves. Paper I also describes the nature of mathematical problem that must be solved numerically: “standing wave boundary conditions” on a large sphere surrounding the sources. In Paper I it is pointed out that there is no a priori meaning to “standing wave boundary conditions” in a nonlinear problem, but that our method requires our standing wave fields to be analogous to the half-outgoing plus half-ingoing solution to a linearized problem. Two methods are given for accomplishing this in a nonlinear problem. One is the iterated Green function method whose results are presented in Paper I. The second method is to define standing waves in the weak-field radiation zone to have equal amplitude of ingoing and of outgoing waves in each multipole, and to have the relative phase of ingoing and outgoing waves adjusted so that the amplitude of the waves is the minimum compatible with the source.

A numerical implementation of this second method requires working with multipoles of the radiation field, and suggests the possibility of using a spectral method for the problem based on multipoles. Some features of the problem argue against this, however. In particular, the sources of the field are compact objects whose size is an order of magnitude less than their separation, therefore requiring a large number of terms in a multipole analysis. Worse, the coupling of source to the field (at least in the black hole case) involves boundary conditions, or matching, on the surface of the compact objects, which would act as an inner boundary for the outer problem. Multipoles, based on coordinates with the topology of spherical polar coordinates are ill-suited to the description of such a surface.

The standard approach to this difficulty is to use coordinate patches and interpolation. In our problem such interpolation may be more difficult to implement than in other boundary value problems. In most problems, involving boundary values for an elliptical problem, relaxation can be used, and interpolation can be done between steps of the relaxation iterations. Our system of partial differential equations, however, is “mixed”; it has both elliptic and hyperbolic regions inside the outer boundary. A consequence is that standard relaxation methods cannot be used. In the most straightforward numerical implementation of interpolation, therefore, the interpolation steps would have to constitute a subset of the set of equations making up the boundary value problem, and would greatly complicate the formulation and numerical solution of the problem.
In this paper we report on an alternative approach, one that has the disadvantage of adding some analytic complexity to the problem, and some worrisome features. But it is a method that gives both remarkably efficient results for model problems, and a potentially useful new approach to the coupling of moving sources to their radiation field. This new method is based on a coordinate system that is adapted to the local structure of the sources and to the large-scale structure of the distant waves. Though the standing-wave boundary condition was the original motivation for introducing an adapted coordinate system, the success with this system suggests that its utility may be more broadly applicable.

This will not be the first application of such coordinates, even in numerical relativity. “ˇCadeˇz coordinates,” [5], a carefully adapted coordinate system of this type, was used in much of the work on head-on collisions of black holes. Like the ˇCadeˇz coordinates, our coordinate systems will reduce to source-centered spherical polar coordinates in the vicinity of the sources, and to rotation-centered spherical polar coordinates far from the sources.

The core of the usefulness of the adapted coordinates is that the field near the sources is well described by a few multipoles in these coordinates, primarily the monopole of the sources, and that the field far from the sources is well described only by a few multipoles in these coordinates. A spectral method (that is, a multipole decomposition), therefore, requires only a small number of multipoles. We will demonstrate, in fact, for mildly relativistic sources (source velocity = 30% c), excellent results are found when we keep only monopole and quadrupole terms.

There is, of course, a price to be paid for this. For one thing, there is additional analytic complexity in the set of equations. Another difficulty is the unavoidable coordinate singularity that is a feature of coordinates adapted to the two different limiting regions. Still, the potential usefulness of the method, and its early successes, have led us to use it not only for the spectral method that is the focus of this paper, but also as the coordinate system for a finite difference method, using the iterated Green function technique reported in Paper I.

B. Nonlinear model problem

The innovative features of this new method present enough new uncertainties that it is important to study this method in the context of the simplest problem possible. We use, therefore, the same model problem as in Paper I, a simple scalar field theory with an adjustable nonlinearity. We will find it quite useful to set the nonlinearity to zero for comparison with the known solution of the linear problem, since many features of our method are unusual even for a linear problem.

For the description of our model problem, we start with Euclidean space coordinatized by the usual spherical coordinates \( r, \theta, \phi \), and we consider sources concentrated near the points \( r = a, \theta = \pi/2 \), in the equatorial plane, and moving symmetrically according to \( \phi = \Omega t \) and \( \phi = \Omega t + \pi \). As in Paper I, we seek a solution of the flat-spacetime scalar field equation

\[
\Psi_{,\alpha;\beta}g^{\alpha\beta} + \lambda F = \nabla^2 \Psi - \partial_t^2 \Psi + \lambda F = \text{Source},
\]

where \( F \) depends nonlinearly on \( \Psi \), in a manner yet to be specified.

We are looking for solutions to Eq. (1) with the same helical symmetry as that of the source motions, that is, solutions for which the Lie derivative \( \mathcal{L}_\xi \Psi \) is zero for the Killing vector \( \xi = \partial_t + \Omega \partial_\phi \). It is useful to introduce the auxiliary coordinate \( \varphi \equiv \phi - \Omega t \). In terms of spacetime coordinates \( t, r, \varphi \) the Killing vector is simply \( \partial_t \) and the symmetry condition becomes the requirement that the scalar field \( \Psi \) is a function only of the variables \( r, \theta \) and \( \varphi \).

(We are assuming, of course, that the form of the nonlinear term is compatible with the helical symmetry.) It is useful to consider the symmetry to be equivalent to the rule

\[
\partial_t \rightarrow -\Omega \partial_\varphi
\]

for scalar functions. In terms of the \( r, \theta, \varphi \) variables, Eq. (1) for \( \Psi(r, \varphi) \) takes the explicit form

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Psi}{\partial \theta} \right) + \left( \frac{1}{r^2 \sin^2 \theta} - \Omega^2 \right) \frac{\partial^2 \Psi}{\partial \varphi^2} + \lambda F(\Psi, r, \varphi) = \text{Source},
\]

that was used in Paper I.

C. Outline

The paper is organized as follows: In Sec. II we introduce the concept of adapted coordinates and some of the details of the specific adapted coordinate system that we will use in this paper. In this section also, we present the
specific form, in these coordinates, of our model nonlinear scalar equation. Some of the details, here and elsewhere, are relegated to appendices. In Sec. III, we explain our spectral method, the expansion of the solution in in a small number of multipoles natural to the adapted coordinates. This method requires very different numerical techniques from those used in the finite difference method, and uses a definition of standing waves that is, at least in principle, from that used in a finite difference approach. Conclusions are given in Sec. IV. Throughout this paper we follow the notation of Paper I[4].

II. ADAPTED COORDINATES

A. Coordinate definitions and properties

For the definition of the adapted coordinates it is useful to introduce several Cartesian coordinate systems. We shall use the notation $x, y, z$ to denote inertial Cartesian systems related to $r, \theta, \phi$ in the usual way (e.g., $z$ is the rotation axis, one of the source points moves as $x = a \cos(\Omega t)$, and so forth). We now introduce a comoving Cartesian system $\tilde{X}, \tilde{Y}, \tilde{Z}$ that is related to $r, \theta, \phi$ in the same way that $x, y, z$ is related to $r, \theta, \phi$, that is, according to

$$\tilde{z} = r \cos \theta \quad \tilde{x} = r \sin \theta \cos (\phi - \Omega t) \quad \tilde{z} = r \sin \theta \sin (\phi - \Omega t).$$

(4)

In this system, as in the inertial $x, y, z$ system, the $\tilde{z}$ axis is the azimuthal axis. We next define the comoving system

$$\tilde{X} = \tilde{y}, \quad \tilde{Y} = \tilde{z}, \quad \tilde{Z} = \tilde{x},$$

(5)

in which the azimuthal $\tilde{z}$ axis is not the rotation axis, but rather the line through the source points, as shown in Fig. 1.

Our goal now is to introduce adapted coordinates, coordinates naturally suited to the physical problem. Far from the source, those coordinates should approach the coordinates most suitable for describing the radiation: spherical coordinates with an origin at the midpoint of the sources. Close to either source point, the coordinates should approach spherical coordinates centered on the source point.

There is, of course, no unique way of defining such coordinates. Here we make the choice that is analytically simplest, though not the choice that is numerically most efficient. Our adapted coordinates $\chi, \Theta, \Phi$ are most easily
FIG. 3: Adapted coordinates in the $\tilde{x}, \tilde{y}$ plane, and three-dimensional coordinate surfaces.

understood, from the distances $r_1$ and $r_2$ from the source points, and the angles shown in Fig. 2. The formal definitions of the adapted coordinates are

$$\chi \equiv \sqrt{r_1r_2} = \left\{ \left[ \left( \tilde{Z} - a \right)^2 + \tilde{X}^2 + \tilde{Y}^2 \right] \left[ \left( \tilde{Z} + a \right)^2 + \tilde{X}^2 + \tilde{Y}^2 \right] \right\}^{1/4} \quad (6)$$

$$\Theta \equiv \frac{1}{2} (\theta_1 + \theta_2) = \frac{1}{2} \tan^{-1} \left( \frac{2\tilde{Z}\sqrt{\tilde{X}\tilde{Y}}}{\tilde{Z}^2 - a^2 - \tilde{X}^2 - \tilde{Y}^2} \right) \quad (7)$$

$$\Phi \equiv \tan^{-1} \left( \frac{\tilde{X}}{\tilde{Y}} \right). \quad (8)$$

This choice is sometimes called “two-center bipolar coordinates”[6], and is equivalent to the zero-order coordinates used by Čadež[5].

The attractive feature of this particular choice of adapted coordinates is that the above relationships can be inverted in simple closed form to give

$$\tilde{Z} = \sqrt{\frac{1}{2} \left[ a^2 + \chi^2 \cos 2\Theta + \sqrt{(a^4 + 2a^2\chi^2 \cos 2\Theta + \chi^4)} \right]} \quad (9)$$

$$\tilde{X} = \sqrt{\frac{1}{2} \left[ -a^2 - \chi^2 \cos 2\Theta + \sqrt{(a^4 + 2a^2\chi^2 \cos 2\Theta + \chi^4)} \right]} \cos \Phi. \quad (10)$$

$$\tilde{Y} = \sqrt{\frac{1}{2} \left[ -a^2 - \chi^2 \cos 2\Theta + \sqrt{(a^4 + 2a^2\chi^2 \cos 2\Theta + \chi^4)} \right]} \sin \Phi. \quad (11)$$

The meaning of the $\chi, \Theta$ coordinates in the $\tilde{x}, \tilde{y}$ plane (the $\tilde{Z}, \tilde{X}$ plane) is shown on the left in Fig. 3; a picture of three-dimensional $\chi, \Theta$, and $\Phi$ surfaces is shown on the right.

The geometrical definition inherent in Fig. 2 suggests that the adapted coordinate surfaces have the correct limit far from the sources. This is confirmed by the limiting forms Eqs. (9)–(11) for $\chi \gg a$. Aside from fractional corrections of order $a^2/\chi^2$ the relations are

$$\tilde{Z} \rightarrow \chi \cos \Theta \quad \tilde{X} \rightarrow \chi \sin \Theta \cos \Phi \quad \tilde{Y} \rightarrow \chi \sin \Theta \sin \Phi. \quad (12)$$

Near the source point at $\tilde{Z} = \pm a$, the limiting forms, aside from fractional corrections of order $\chi^2/a^2$ are

$$\tilde{Z} \rightarrow \pm a + \frac{\chi^2}{2a} \cos (2\Theta) \quad \tilde{X} \rightarrow \frac{\chi^2}{2a} \sin (2\Theta) \cos \Phi \quad \tilde{Y} \rightarrow \frac{\chi^2}{2a} \sin (2\Theta) \sin \Phi. \quad (13)$$

These limits, and Fig. 2 show that near the source point at $\tilde{Z} = a$ the expression $\chi^2/2a$ plays the role of radial distance, and $2\Theta$ plays the role of polar coordinate. (Near the source point at $\tilde{Z} = -a$, the expression $\chi^2/2a$ again plays the role of radius, but $\pi - 2\Theta$ is polar angle.) Notice that both for the near and the far limit, the polar angle is defined with respect to the line through the sources, the $\tilde{Z}$ axis, not with respect to the rotational $\tilde{z}$ axis.

It is clear that our new system has a coordinate singularity at the origin. Indeed, there must be a coordinate singularity in any similar adapted coordinate system. The switch from the small-$\chi$ coordinate surfaces, disjoint 2 spheres around the sources, to the large-$\chi$ single 2-sphere cannot avoid a singularity.
B. Nonlinear scalar model in adapted coordinates

In terms of the comoving Cartesian coordinates, the helical symmetry rule in Eq. (2) takes the form
\[
\partial_t \rightarrow -\Omega \left( \frac{x}{\partial y} - \frac{y}{\partial x} \right) = -\Omega \left( \frac{\tilde{Z}}{\partial \tilde{X}} - \frac{\tilde{X}}{\partial \tilde{Z}} \right).
\] (14)

Our nonlinear scalar field equation of Eq. (1) can then be written, for helical symmetry, as
\[
\mathcal{L}\Phi + \lambda F = \frac{\partial^2 \Psi}{\partial \chi^2} + \frac{\partial^2 \Psi}{\partial \Theta^2} + \frac{\partial^2 \Psi}{\partial \Phi^2} + \Omega^2 \left( \frac{\tilde{Z}}{\partial \tilde{X}} - \frac{\tilde{X}}{\partial \tilde{Z}} \right)^2 \Phi + \lambda F = \text{Source}.
\] (15)

With the relations, given in Eqs. (9)–(11), between the \( \tilde{X}, \tilde{Y}, \tilde{Z} \) system and the three-dimensional adapted coordinates, the field equation can be expressed completely in terms of adapted coordinates in the form
\[
\mathcal{L}\Phi + \lambda F = A_{\chi\chi} \frac{\partial^2 \Psi}{\partial \chi^2} + A_{\chi\Theta} \frac{\partial^2 \Psi}{\partial \Theta^2} + A_{\chi\Phi} \frac{\partial^2 \Psi}{\partial \Phi^2} + 2A_{\chi\Theta} \frac{\partial^2 \Psi}{\partial \chi \partial \Theta} + 2A_{\chi\Phi} \frac{\partial^2 \Psi}{\partial \chi \partial \Phi} + 2A_{\Theta\Phi} \frac{\partial^2 \Psi}{\partial \Theta \partial \Phi}
\]
\[
+ B_{\chi} \frac{\partial \Psi}{\partial \chi} + B_{\Theta} \frac{\partial \Psi}{\partial \Theta} + B_{\Phi} \frac{\partial \Psi}{\partial \Phi} + \lambda F = \text{Sources}.
\] (16)

The \( A_{ij} \) and \( B_i \) coefficients here are explicitly known functions of \( \chi, \Theta, \Phi \) found from the partial derivatives of the \( \tilde{X}, \tilde{Y}, \tilde{Z} \) with respect to the adapted coordinates. The explicit values of these coefficient functions are given in Appendix A.

We use the same nonlinear term that we did in Paper I
\[
\lambda F = \lambda \left( \frac{\chi}{na} \right)^n e^{(n-\chi/a)} \frac{\Psi^3}{1 + \Psi^2}.
\] (17)

The parameter \( \lambda \) allows us to adjust the importance of the nonlinearity, and to turn it off to explore linear solutions. The prefactor \( (\chi/na)^n \exp(n - \chi/a) \) in \( F \) contains a parameter \( n \) that allows us to concentrate the nonlinearity in different locations. The prefactor has a maximum of unit at \( \chi = na \), and falls off quickly for \( |\chi - na| \) larger than \( a \).

To identify a physical problem, we must specify the source term, or its equivalent, in Eq. (16). In Paper I we used point-like source terms. With our adapted coordinates we can formulate the problem in a more physical way: We set the source to zero, and we supply inner boundary conditions at the approximate spherical surface for some \( \chi = \chi_0 \), where \( \chi/a \) is small.

The remaining specification needed is the outer boundary conditions on some large approximate spherical surface \( \chi = \chi_{\text{max}} \). The usual Sommerfeld outgoing outer boundary condition \( \partial_r \psi = -\partial_r \psi \), can be approximated as \( \partial_r \psi = -\partial_r \psi \). The fractional error introduced by this substitution is of order \( a^2/\chi^2 \). The Sommerfeld condition itself is accurate only up to order wavelength/\( r \). Since the wavelength is larger than \( a \), our substitution \( r \rightarrow \chi \) in the outer boundary condition introduces negligibly small errors. To apply the helical symmetry we use the replacement rule in Eq. (14) and the outgoing boundary condition becomes
\[
\frac{\partial \Psi}{\partial \chi} = \Omega \left( \frac{\tilde{Z} \partial \Psi}{\partial \tilde{X}} - \frac{\tilde{X} \partial \Psi}{\partial \tilde{Z}} \right) = \Omega \left( \Gamma_\Theta \frac{\partial \Psi}{\partial \Theta} + \Gamma_\Phi \frac{\partial \Psi}{\partial \Phi} + \Gamma_\chi \frac{\partial \Psi}{\partial \chi} \right).
\] (18)

where the \( \Gamma_s \) are given explicitly in Appendix A. At large \( \chi \) the outgoing condition can be written
\[
\frac{\partial \Psi}{\partial \chi} = \Omega \left( \frac{\cos \Phi \partial \Psi}{\partial \Theta} - \frac{\cos \Theta \sin \Phi \partial \Psi}{\partial \Phi} \right) \left( 1 + \mathcal{O}(a^2/\chi^2) \right).
\] (19)

The correction on the right is higher-order at the outer boundary \( \chi = \chi_{\text{max}} \) and can be ignored. The ingoing boundary condition follows by changing the sign of the right hand side of Eq. (18) or (19).

The problem of Eqs. (16) (17) and (19) is a well-posed boundary-value problem analogous to that in Paper I[4]. As in Paper I, this problem can be numerically implemented using the finite difference method (FDM) of discretizing derivatives. The difference between such a computation and that of Paper I is, in principle, only in the coordinate dependence of the coefficients \( (A_{\chi\chi}, A_{\Theta\Theta}, \cdots) \) in the differential equation and the outer boundary condition.

We have carried out such a computation, but with an additional difference from the computation in Paper I. Rather than include an explicit, inhomogeneous source term in the field equation (16), we have used Dirichlet data...
on a constant-χ surface χ = χ₀ ≪ a. This data has been chosen to be that for the linearized solution from two point sources. That solution is expressible in closed form as a series involving spherical harmonics and spherical Bessel functions of r, θ, φ coordinates⁴. Since these coordinates can be found from χ, Θ, Φ with Eqs. (4), (5) and (9)-(11), we can compute the “exact” linearized Ψ at χ = χ₀, and use this as inner boundary data. This approach is more appropriate for the eventual application to black hole sources; in Paper I an explicit source was used because inner boundary conditions would have been difficult to impose in spherical coordinates.

Aside from the differences already mentioned, our FDM computations repeated the numerical techniques (direct matrix inversion, Newton-Raphson iteration, continuation) of Paper I, and are not presented here. Two points, however, are worth mentioning. First, there were no fundamental difficulties in computing solutions; the only difficulties were essentially the same difficulties encountered with the FDM using spherical coordinates in Paper I. This expected result confirmed the fact that there is nothing ill-posed in the boundary value problem in the adapted coordinates.

The second point worth mentioning concerns a feature of the problem that may indeed seem ill-posed: the coordinate singularity at χ = a, Θ = Π/2 (see Fig. 3). We have found that the truncation error at these points dominates the computational error, and causes the code to be only first-order convergent. A study has been completed of how to remove this dominant error, and restore second-order convergence, in a relatively simple way, by introducing finite elements in the region of the singularity⁷.

**NOTE: WE NEVER REALLY WORKED OUT THE FORM NEAR THE SINGULARITIES.**

### III. SPECTRAL SOLUTIONS

#### A. Spectral decomposition

A second approach to solving the nonlinear problem, fundamentally different from the finite difference method, is to expand Ψ in a complete set of functions of the angular coordinates Θ and Φ. Since one of our goals is to describe the radiation in the weak wave zone, and since Θ and Φ approach comoving spherical coordinates in the weak wave zone, the natural set of basis functions is the spherical harmonics Y_ℓm(Θ, Φ). In terms of these we would look for a solution of Eq. (16) in the form

$$\Psi = \sum_{\text{even } \ell} \sum_{m} a_{\ell m}(\chi) Y_{\ell m}(\Theta, \Phi).$$

(The odd ℓs are omitted due to the symmetry of the problem.)

The primary motivation for this kind of spectral approach was explained in Paper I. One of our definitions of nonlinear standing waves must be imposed on individual multipoles. In particular, we use the condition that in the weak wave zone (where linearized approximations apply): (i) the amplitude of the ingoing and outgoing radiation in each multipole are equal, and (ii) the relative phase of the ingoing and outgoing parts in each multipole are such that the amplitude of the radiation is minimized consistent with coupling to the source. (Such minimization is a feature of the half-ingoing, half-outgoing standing waves of linear theory.)

These multipole-by-multipole conditions must be satisfied as part of the set of equations that constitute the boundary value problem. If those equations describe projection of the multipole components, they add great complexity to the set of equations that must be solved. This minimization method for standing waves, therefore, is best implemented with a numerical method that works directly with radiation multipoles.

We shall limit ourselves in Eq. (20) only to the ℓ = 0 and ℓ = 2 modes for two reasons. First, considerable work needs to be done to add more modes. Second, and more important, we find impressively accurate results just keeping this minimal set of modes. The symmetry of the physical problem precludes the scalar field having a terms proportional to sin Φ or sin 2Φ, so a full set of normalized, real ℓ = 2, m = 0, ±1, ±2 angular functions for the scalar field is

$$Y_2 \equiv Y_{20}(\Theta, \Phi) = \frac{1}{2} \sqrt{\frac{5}{4\pi}} (3\cos^2 \Theta - 1)$$

$$Y_C \equiv \sqrt{2} \text{Re}(Y_{22}(\Theta, \Phi)) = \frac{1}{4} \sqrt{\frac{15}{\pi}} \sin^2 \Theta \cos 2\Phi$$

$$Y_S \equiv - \sqrt{2} \text{Re}(Y_{21}(\Theta, \Phi)) = \frac{1}{4} \sqrt{\frac{15}{\pi}} \sin 2\Theta \cos \Phi.$$  

Note that the 2, C and S modes correspond to the quadrupole moments of the field at large χ. Near the sources at small χ, however, Θ = θ₁/2 or Θ = θ₁/2 + π/2. (See Fig. 2.) The 2, C and S modes near the source, therefore, correspond to a mixture of dipole and higher multipole details of the source.
It must be remembered that for \( \chi \gg a \) the adapted coordinate angles \( \Theta, \Phi \) approach spherical coordinates with respect to the \( \bar{x}, \bar{y}, \bar{z} \) axes, and the azimuthal axis \( \bar{z} \) is not the rotation axis. The radiation is best described with the comoving angular coordinates \( \theta, \varphi \) related to the \( \bar{x}, \bar{y}, \bar{z} \) axes. Because of this, we do not directly use the functions in Eq. (21), but rather we use the linear combinations

\[
Y_2 = -\frac{1}{2} Y_2 + \sqrt{\frac{3}{2}} Y_C = \frac{1}{8} \sqrt{\frac{5}{\pi}} (1 - 3 \cos^2 \Theta + 3 \sin^2 \Theta \cos 2\Phi)
\]

\[
Y_C = \frac{\sqrt{3}}{2} Y_2 - \frac{1}{2} Y_C = \frac{1}{8} \sqrt{\frac{15}{\pi}} (3 \cos^2 \Theta - 1 - \sin^2 \Theta \cos 2\Phi)
\]

\[
Y_S = Y_S = \frac{1}{4} \sqrt{\frac{15}{\pi}} \sin 2\Theta \cos \Phi.
\]

The addition theorem for spherical harmonics guarantees that \( \ell = 2 \) spherical harmonics with respect to the \( \bar{x}, \bar{y}, \bar{z} \) axes can be expressed as linear combinations of the \( \ell = 2 \) spherical harmonics with respect to the \( x, y, z \) axes. When the transformation is explicitly carried out one finds

\[
Y_2, Y_C, Y_S \equiv \begin{cases} 
\frac{1}{2} \sqrt{\frac{3}{\pi}} (3 \cos^2 \theta - 1) = Y_{20}(\theta, \varphi) \\
\frac{1}{4} \sqrt{\frac{15}{\pi}} \sin^2 \theta \cos 2\varphi = \sqrt{2} \Re \{Y_{22}(\theta, \varphi)\} \\
\frac{1}{4} \sqrt{\frac{15}{\pi}} \sin^2 \theta \sin 2\varphi = \sqrt{2} \Im \{Y_{22}(\theta, \varphi)\}.
\end{cases}
\]  

(23)

The real, normalized set of functions of \( \theta, \varphi \) in Eq. (23) is complete for describing our \( \ell = 2 \) scalar fields with respect to the \( x, y, z \) axes. (Note that the missing \( m = 1 \) component in this system is consistent with the symmetry of the source and field.)

With the functions in Eq. (22), our \( \ell = 0, 2 \) approximation to the scalar field is then written

\[
\Psi = a_0(\chi) Y_0 + a_2(\chi) Y_2(\Theta, \Phi) + a_C(\chi) Y_C(\Theta, \Phi) + a_S(\chi) Y_S(\Theta, \Phi)
\]

(24)

where \( Y_0 \equiv Y_{00} = 1/\sqrt{4\pi} \). This approximation is then substituted in Eq. (16) to give

\[
\sum_{k=0,2,C,S} \frac{d^2 a_k}{d\chi^2} [A_{k\chi} Y_k] = - \sum_{k=0,2,C,S} \left\{ a_k(\chi) \left[ A_{\Theta \Theta} \frac{d^2 Y_k}{d\Theta^2} + A_{\Phi \Phi} \frac{d^2 Y_k}{d\Phi^2} + 2A_{\Theta \Phi} \frac{d^2 Y_k}{d\Theta d\Phi} + B_{\phi} \frac{d Y_k}{d\Theta} + B_\phi \frac{d Y_k}{d\Phi} \right] \right\} - \lambda F \left( \chi, \Theta, \sum_k a_k(\chi) Y_k(\Theta, \Phi) \right).
\]

(25)

We must next project out from Eq. (25) a set of ordinary differential equations, via an integration \( \int \int Y_p W d\Theta d\Phi \), where \( W \) is a weight function used in the projection. It is useful to put the resulting equations in the form

\[
\sum_{k=0,2,C,S} \alpha_{pk} \frac{d^2 a_k}{d\chi^2} = R_p(\chi, a_q, da_q/d\chi) \quad p = 0, 2, C, S,
\]

(26)

in which the \( \alpha_{pk} \) are explicitly known functions of \( \chi \) and the “source” terms \( R_p \) are functions of \( \chi \), and of all the \( a_k \) and their \( \chi \) derivatives. For linear models, the \( R_p \) right-hand sides are linear in \( a_k \) and \( da_k/d\chi \). Details are given in Appendix B.

There are several considerations that must be taken into account in the choice of the weight function \( W \). For the most convenient description of the radiation field we want the multipoles asymptotically to decouple in the weak wave zone. This requires that

\[
W \chi \rightarrow \sin \Theta,
\]

(27)

the usual weight function on the sphere. (In principle, we could also use \( \sin \theta \) or the sine of any other angle with respect to a corotating Cartesian system.)
A second consideration in the choice of \( W \) is closely related to a feature, illustrated in Fig. 4, of the constant-\( \chi \) surfaces, for \( \chi < a \). For \( \chi < a \) the coordinates \( \chi \) and \( \Theta = \pi/2 \) specify a point. But the angular functions \( Y_2^m \) and \( Y_C^m \) have nonzero \( \Phi \) variations at \( \Theta = \pi/2 \); these functions are therefore multivalued at \( \chi < a, \pi/2 \), and hence the spectral sum in Eq. (24) is multivalued at such points. In view of this, it is important to reconsider the meaning the approximation in Eq. (24).

At a constant \( \chi < a \) the scalar field \( \Psi \) is a single valued function of \( \Theta \), \( \Phi \) in the range \( 0 \leq \Theta \leq \pi \) and \( 0 \leq \Phi \leq 2\pi \). The spherical harmonics are complete in this range, in the usual sense of convergence in the mean. It follows that \( \Psi \) can be usefully expanded in a series of spherical harmonics of \( \Theta, \Phi \), but that series will not necessarily exhibit pointwise convergence at the troublesome \( \chi < a, \pi/2 \) points. Our approximation in Eq. (24) should be viewed with this understanding. The representation of \( \Psi \) near \( \chi < a, \Theta = \pi/2 \) may be highly inaccurate, but the approximation may be quite accurate for other purposes, in particular for linking the sources to the distant radiation field. The numerical results below will be evidence that this is, in fact, the case.

The difficulties at \( \chi < a, \Theta = \pi/2 \) reflect themselves in singularities in the integrals that arise in projecting out multipole components. In particular, the difficulty is associated with the factor \( Q - a^2 - \chi^2 \cos 2\Theta \) in the denominator of several of the integrals. (See terms in Eq. (B2) associated with \( A_{\Phi\Phi} \) and \( B_{\Phi} \).) At \( \Theta = \pi/2 \) this factor is finite for \( \chi > a \), but for \( \chi \leq a \) behaves as

\[
Q - a^2 - \chi^2 \cos 2\Theta \approx \frac{2 \chi^2}{a^2 - \chi^2} \cos^2 \Theta.
\] (28)

To meet the requirements on our weight function we start by defining

\[
W_1 = Q - a^2 - \chi^2 \cos 2\Theta,
\] (29)

which cancels the divergent behavior in the integrals, and

\[
W_2 = \sqrt{\frac{Q - a^2 - \chi^2 \cos 2\Theta}{2}},
\] (30)

which approaches \( \chi \sin \Theta \) for \( \chi \gg a \). We then take our weight function \( W \) to be \( W_1 \) for \( \chi < \chi_1 > a \), and to be \( W_2 \) for \( \chi > \chi_2 > \chi_1 \). In the intermediate region, between \( \chi_1 \) and \( \chi_2 \), we use

\[
W = \gamma(\chi)W_2 + [1 - \gamma(\chi)]W_1
\] (31)

where

\[
\gamma(\chi) \equiv 3 \left[ \frac{\chi - \chi_1}{\chi_2 - \chi_1} \right]^2 - 2 \left[ \frac{\chi - \chi_1}{\chi_2 - \chi_1} \right]^2.
\] (32)

increases from 0 at \( \chi_1 \) to unity at \( \chi_2 \) and has vanishing derivative at both endpoints.

Since the weight function is independent of \( \Phi \), the \( \Phi \) dependence is very simple in the integrands for the \( \alpha_{pk} \) [see Eq. (B1)]. The \( \Phi \) integration can be carried out analytically, thereby reducing the numerical evaluation of \( \alpha_{pk} \) to a single integral at each value of \( \chi \). This cannot, in general, be done for \( R_p \), due to the nonlinear dependence of \( R_p \) on the \( a_k \).

**B. Outer boundary conditions**

The outgoing boundary conditions to be used with Eq. (16) are \( \partial_t \Psi = -\partial_r \Psi \); the ingoing boundary condition uses the opposite sign. As in Eq. (18), we replace this by \( \partial_t \Psi = -\partial_\chi \Psi \) at the outer boundary \( \chi_{\text{max}} \), with no loss of
accuracy. We can apply this outgoing condition to the expansion in Eq. (24) to get

\[ da_C/d\chi = 2\Omega a_S \quad da_S/d\chi = -2\Omega a_C, \] (33)

at \( \chi = \chi_{\text{max}} \). At this level of large-\( \chi \) approximation Eq. (26) gives \( d^2a_k = -4\Omega^2 a_k \) for \( k = C, S \), so the two conditions in Eq. (33) are not independent.

The functions \( a_0 \), and \( a_2 \) are not oscillatory, and fall off at large \( \chi \) as \( \chi^{-1} \) and \( \chi^{-3} \), respectively. For these functions, then we use the boundary conditions

\[ da_0/d\chi = -a_0/\chi \quad da_2/d\chi = -3a_2/\chi \] (34)

at \( \chi_{\text{max}} \).

The standing wave problem is in some ways simpler than the outgoing problem. The symmetry of the standing wave problem requires that Eq. (24) be replaced by

\[ \Psi = a_0(\chi) \, \mathcal{Y}_0 + a_2(\chi) \, \mathcal{Y}_2(\Theta, \Phi) + a_C(\chi) \, \mathcal{Y}_C(\Theta, \Phi). \] (35)

There is no boundary condition analogous to Eq. (33), and any solution will be a standing wave solution. (It will have an equal mixture of ingoing and outgoing radiation in each multipole mode.) The numerical task is to search for the standing wave solution with the minimum amplitude.

C. linear problems: numerical methods and results

If the nonlinearity is removed from the problem, by setting \( \lambda = 0 \), the computation is greatly simplified. Due to the simplification, and the availability of series solutions, the linear problem is a useful way of checking the fundamental correctness of the spectral approach.

The first simplification in the linear case is that Eq. (26) can be put in the form

\[ \sum \alpha_{pk} \frac{d^2a_k}{d\chi^2} = \sum R_{pk}^{(1)} \frac{da_k}{d\chi} + \sum R_{pk}^{(2)} \, a_k. \] (36)

Here the coefficients \( \alpha_{pk} \), \( R_{pk}^{(1)} \), and \( R_{pk}^{(2)} \) are functions of \( \chi \), that can be evaluated independently of the values of the \( a_k(\chi) \). The second simplification in the linear case is that the \( \Phi \) integration in \( R_{pk}^{(1)} \), and \( R_{pk}^{(2)} \), like that for the \( \alpha_{pk} \) can be carried out analytically. All angular integrals, therefore, can be reduced to single integrals, and can be done at the outset, independently of the process of solving for the \( a_k \). We perform that \( \Theta \) integration using a “smart” Gaussian code that selects among possible Jacobian polynomials for the most appropriate. Relative error in the integration is set at ??? Here we needn’t dwell on errors... after all, we’re comparing with the exact solution, but we should make some statement... maybe a statement about how large an integration error we can use.

It should be noted that for the ingoing, or outgoing problem we are free to choose eight independent values of inner boundary data \( a_k \) and \( da_k/d\chi \), for \( k = 0, 2, C, S \) at \( \chi_0 \), and we need to satisfy only three outer boundary conditions at \( \chi_{\text{max}} \). In practice, we fix \( a_0 \) and \( da_0/d\chi \) at \( \chi_0 \) to correspond to the local scalar charge monopole field, leaving three degrees of freedom for the specification of \( a_2 \), \( a_C \), \( a_S \), \( da_2/d\chi \), \( da_C/d\chi \), \( da_S/d\chi \), consistent with the outer boundary conditions. How are these three degrees of freedom to be fixed? As long as the outgoing solution in Eq. (33), or its ingoing equivalent, is satisfied, we find that the solution we compute is insensitive to a wide range of choices within these three degrees of freedom.

The physical explanation of this lies in the meaning of the inner boundary specifications. At large \( \chi \) the character of the waves (amplitude and phase of the ingoing and outgoing waves) is very sensitive to to the full set of specifications. A small change of the inner boundary data can change the waves from ingoing to outgoing. From a physical point of view the full set of specifications contains information about the detailed angular structure of the sources, in particular the local dipole and quadrupole moments. This information should not be an important determinant of the distant outgoing waves; those waves are a result of the circular motion of the pair of monopole sources. This physical fact is represented in Eq. (lin2Bsolved) by the way in which \( a_0 \) acts as a source for \( a_C \) and \( a_S \) in the outward \( \chi \) integration. The values of \( a_C \) and \( a_S \) at \( \chi_0 \) are, in principle, fine details of the sources, details that are not of primary importance to the determination of the distant waves. In practice, at \( \chi_0 \) the “monopole” \( a_0 \) does not carry all the monopole information, so the initial data for the \( C \) and \( S \) modes must be specified, but we need only arrange for them to give purely outgoing (or ingoing) waves. Other details of these modes encode the fine angular details of the source. In principle, then, we more-or-less arbitrarily fix three values of \( a_2 \), \( a_C \), \( a_S \), \( da_2/d\chi \), \( da_C/d\chi \), \( da_S/d\chi \), and we adjust the
other three to satisfy the large-$\chi$ conditions. The results (as we shall demonstrate) are insensitive to our arbitrary choice of the three fixed values. (There is, of course, a limit to this insensitivity. Multipoles should be physically due to distortions of the sources and hence can be no larger than the monopole moment times the $\ell$th power of the size of the source $\chi_0$. If we take $a_2$, $a_C$, or $a_S$ to be much larger than $a_0\chi_0$, equivalent to invoking a “point” scalar dipole at the source, then the fields at large $\chi$ will be significantly affected.)

For the linear case, the problem of inner data is somewhat simplified. The coefficients $A_{0\Theta} \cdots B_{\chi}$ in Eq. (25) have parts that are proportional to $\exp(in\Phi)$ with $n = 0$ or 2; there is no $n = 1$ part. This means that the $a_S$ problem does not mix with that of and $a_2$, $a_C$.

There are two rather different numerical approaches that can be used to find a solution. The first is a shooting method. A differential equation solver is started at $\chi_0$ and used to solve out to $\chi_{\text{max}}$; we have done this with a fourth-order Runge-Kutta algorithm both with fixed step size and adaptive step size. To solve the $a_S$ problem, in this method we arbitrarily fix $a_0$ at $\chi_0$; the value of $\chi_{\text{max}}$ at $\chi_0$ is determined by the outgoing boundary condition in conjunction with the solution for $ac$. In the $a_0$, $a_2$, $ac$ sector, the value of $a_0$ and $\chi_{\text{max}}$ at $\chi_0$ is fixed as described above. The value of $a_2$ and $\chi_{\text{max}}$ at $\chi_0$ are fixed to correspond to a local finite static field. Since $a_2$ does not couple to radiation, this is sufficient to ensure that $a_2$ satisfies the outer boundary condition in Eq. (34). We then choose arbitrary values of $ac$ and $\chi_{\text{max}}$.

With the starting values, at $\chi_0$, of all $a_k$ and $\chi_{\text{max}}$ set, we then integrate out to $\chi_{\text{max}}$ both for the $a_S$ and the $a_0$, $a_2$, $ac$ sector, and we calculate, at $\chi_{\text{max}}$ the value of

$$test_1 = da_S/\chi + 2\Omega a_C.$$ 

We next change our guess for $da_S/\chi$ at $\chi_0$, and solve in the $a_S$ sector again, and we compute a quantity $test_2$ at $\chi_{\text{max}}$ as above. Some linear combination $c_1test_1 + (1 - c)test_2$ will vanish, and we take as our solution for $a_S$ this same linear combination of the first and second trial solutions for $a_S$.

An alternative way of solving is to treat the determination of $a_k(\chi)$ as a boundary value problem. At a discrete set of $N$ points $\chi_j$, with $\chi_N = \chi_{\text{max}}$, the values of $a_k(\chi_j)$ are treated as unknowns. The four differential equations of Eq. (36) are implemented as $4(N - 1)$ second-order correct finite difference representations at $\chi_j$, with $j = 1 \cdots N - 1$. We specify, as before, the values of $a_0$, $a_2$, $ac$ and $a_S$ at $\chi_0$, and we fix $\chi_{\text{max}}$ at $\chi_0$. The finite difference equation for $\chi_{\text{max}}$ along with the three outer boundary conditions at $\chi_N = \chi_{\text{max}}$ gives us the four additional equations needed to determine the $4N$ unknowns. The resulting set of $4N$ linear equations are solved by a standard matrix inversion algorithm.

The results of computing outgoing waves are shown in Fig. 5 for three values of source velocity $a\Omega$. The boundary-value method of determining $a_k(\chi)$ was used for these graphs, and was found to work somewhat more stably than the shooting method. (The difference between the two methods is on the order of 3%, which is around the level of numerical error due to truncation and angular integration.) The thick solid curve shows the result of our $\ell = 0, 2$ spectral method with the choices that $a_2$, $\chi_{\text{max}}$, $ac$, $a_C$ vanish at $\chi_0 = 0.3$. The outgoing outer boundary condition is imposed at $\chi_{\text{max}} = 30\ a$. The variation in the result is less than 2%, when the grid size in $\Theta$ or $\chi$ is doubled. Put here more info on computing. We notice that the accuracy of the spectral method is marginal for $a\Omega = 0.4$.

The error arises not from the numerical methods (grid resolution, round-off) themselves but rather from the inclusion of only $\ell = 0$ and $\ell = 1$ modes. The amplitude of the waves of each multipole, relative to those of another multipole, are of order $\Omega^\ell$. As $\Omega$ increases the high-$\ell$ modes become more and more important. For $a\Omega = 0.4$, it is the omission of the $\ell = 4, 6, \cdots$ that accounts for the large error.

Figure 6 illustrates the insensitivity of the solution to the inner boundary data. Results are given here for $a\Omega = 0.3$, with $\chi_0 = 0.3\ a$ and $\chi_{\text{max}} = 30\ a$. Results are given for three different values of $ac$, a value of 0 (solid curve), a value of $0.1\ a^2$ (dashed curve), and a value of $0.2\ a^2$ (dotted curve). The difference between the Runge-Kutta method and the matrix inversion method are negligible, less than 2%. As explained above, the different choices of $ac$ affects only the structure of the field very near the sources.

In the finite difference method of Paper I, the “linear standing wave problem” did not exist. In that method, standing waves were defined by an iterative process. For the linear problem, only the first step in the iteration exists, and that step is to take half the sum of the ingoing and outgoing wave solutions. The method, then is guaranteed to give the correct (half-in, half-out) result for the linear problem. This is not the case in our spectral method. Here the definition of standing wave is minimum wave amplitude for a radiation-balanced solution, and even in the linear case there is no guarantee that our spectral method will produce the correct half-in, half-out solution. The computation, then, is a useful test of the method.

The computational problem for standing waves only the $k = 0, 2$ and $C$ modes are involved, and the only outer boundary conditions are those of Eq. (33). In practice, we specify $a_0$ and $\chi_0/\chi$ at $\chi_0$ as previously for outgoing or ingoing waves. We then fix, at $\chi_0$, three of the four values $a_2$, $\chi_{\text{max}}$, $ac$, $a_C/\chi$ and we shoot outwards with
various choices of the fourth value to find the solution with the minimum wave amplitude at large $\chi$. To evaluate the amplitude, we note that the large $\chi$ waves will have the form

$$\Psi = A \cos 2\Omega \chi + \delta$$

where we have approximated $r \approx \chi$. From the computed solution we construct the quantity

$$\text{SqAmp} \equiv \frac{1}{4\Omega^2 \chi^2} \left[ \chi \left( \frac{d\Psi}{d\chi} + \Psi \right)^2 + [\chi \Psi]^2 \right], \quad (37)$$

which, at large $\chi$ is roughly independent of $\chi$ and is a measure of the square amplitude of the standing wave. Note that only the shooting method can be used for the standing wave problem. There is not a complete set of outer boundary conditions, so the boundary-value method is not appropriate.

Results are given in Fig. 7 for the computation of the linear standing waves. These results are seen to be in excellent agreement with the exact (series solution), demonstrating that at least in the linear case the minimization method with the adapted-coordinate/spectral method is sound, and is accurate.

Results in the present paper are given only for three-dimensional models, but it is worth pointing out that for the linear two-dimensional equivalent of the problem here, the computational problem simplifies due to features other than a simple reduction of dimensionality. The relative simplicity of the problem makes it feasible to include a large number of multipoles, and to explore other basic issues in the spectral method with adapted coordinates. For that reason, the formalism for standing waves in the two-dimensional linear problem is given in Appendix C.
D. nonlinear problems: numerical methods and results

If the shooting method is used to solve for outgoing, ingoing, or standing waves, the method for nonlinear problems differs only in a few details from the linear method. For the outgoing, or ingoing solutions the nonlinearity couples the $S$ sector with $0, 2, C$ sector, and we can no longer use a linear combination of two runs to satisfy outer boundary conditions. In practice, this does not add great difficulty. For the standing wave solution the procedure for finding a minimum-amplitude solution follows precisely the same steps as for the linear problem.

For the outgoing or ingoing solutions (but not for the standing wave solution), the boundary value method is an alternative to shooting. For the nonlinear problem the set of $4N$ equations is nonlinear. These equations are solved with a Newton-Raphson iteration, and with continuation in $\lambda$ or $\Omega$, much as in the manner of the FDM solutions of Paper I.

The most important question about the nonlinear solutions is whether effective linearity applies, that is, whether a good approximation for the nonlinear outgoing solution can be extracted from the nonlinear standing wave solution. We have seen that to be the case for the numerical results from our FDM solutions in Paper I, but it is an important question to ask again with the adapted-coordinate/spectral method. Here, not only is the numerical implementation of the problem completely different from the FDM approach, but the minimum amplitude criterion for standing waves is fundamentally different from the iterative definition in Paper I. We present the answer to this question in Fig. 8. That figure shows the true nonlinear outgoing solution and the extracted outgoing solution for strongly nonlinear models. For comparison, those plots also show the equivalent linear models to illustrate the strength of the nonlinearity. They also include the results from the same models from the FDM computations of Paper I.

This is a hope, and might be a fantasy. It would be wonderful to show a meaningful comparison of the “same” Paper I and Paper II models, but life is not simple. For one thing, we may have trouble getting results with nonlinearities as strong as those of Paper I. More important, unless $\chi_0 \rightarrow 0$ there’s really no way of having the “same” nonlinear model.

The numerical uncertainties in the adapted-coordinate/spectral results in Fig. 8 arise in several ways. Once source is the truncation error in the finite difference representation of the ordinary differential equations of Eq. (36), and of the boundary conditions. By varying the grid size we have confirmed that the solution is second order (????) and that the truncation error in the results shown are smaller than 2%. The true outgoing solution shown in Fig. 8 was found with the boundary value method, and this is subject to the error in the iterative matrix inversion procedure. That error is controlled by error flags set in the inversion routine, and we have confirmed that it is negligible compared to the truncation error.

The nonlinearities in the models of Fig. 8 are $\lambda = ?$ for the $n = 0$ case, and $\lambda = ?$ for $n = 5$. The strength of the nonlinearity is affected both by $\lambda$ and by $n$, so the most meaningful indication of the strength of the nonlinearity is the difference between the linear and nonlinear solutions.

The results in Fig. 8 show, as did those in Paper I, that effective linearity is very effective indeed. The extracted outgoing solution is an excellent approximation to the true outgoing solution. Indeed, the difference between the two in the $n = 0$ case is on the order of the numerical uncertainty in the numerical determination. As explained in Paper I, the success of effective linearity is rooted in the fact that the nonlinearities are confined to regions well away from the wave zone. As we did in Paper I we probe this in the $n = 5$ model by artificially moving the region of nonlinearity outward into the wave zone. As expected, and as found in Paper I, this does break effective nonlinearity and induce
a significant difference between the extracted outgoing solution and the true outgoing solution.

An important result that can be seen in Fig. 8 is that for a given model we get the same result from the FDM method of Paper I, and the adapted-coordinate/spectral method of the present paper. This is particularly important because the definitions of nonlinear standing waves is different in the two papers. Had we found important differences in the computed standing waves from the two definitions it would have suggested that our results are artifacts of our definitions. It is important to understand, however, that the insensitivity of the standing waves to the difference in definition is not logically independent of the success of effective linearity. The meaning of the true outgoing solution does not depend on choice or definition. If, then, the two nonlinear standing wave solution, for the two definitions, each contain a good approximation to the true outgoing solution, the two nonlinear standing wave solutions must be in good agreement.

IV. CONCLUSIONS

Thoughts: conclusions should be pretty much limited to adapted coordinates.

- The adapted coordinates lead to a well-posed boundary value problem. These coordinates are well suited to the specification of inner boundary conditions rather than an explicit source term. The boundary value problem can be solved by FDM as in Paper I.

- The adapted coordinates allow us to use an unusual spectral method which seems to encode the necessary information with remarkable efficiency, and to be a mathematical embodiment of the essence of radiation generation. One facet of this approach is the demonstrated insensitivity of the results to most of the details at the inner boundary.

- There may be better adapted coordinates. Details about 2D linear and solution of harmonic, but in GR.

- For nonlinear problems the spectral method is well suited to a definition of standing waves that is fundamentally different from the iterative definition used with the FDM method.

- The adapted/spectral method has very low RAM requirements.

- The time consuming step, the computational bottleneck, is the angular integrations. There may be all sorts of tricks for improving these angular integrations.

- There is insignificant difference found in the results of the two (minimization vs iterative) definitions of nonlinear standing waves.
APPENDIX A: COEFFICIENTS FOR THE THREE-DIMENSIONAL FIELD EQUATION

The transformation from Eq. (15) to Eq. (16) leads to the expressions

\[ A_{\chi\chi} = \vec{\nabla}_{\chi} \cdot \vec{\nabla}_{\chi} - \Omega^2 \bar{A}_{\chi\chi} \]  \hspace{1cm} (A1)

\[ A_{\Theta\Theta} = \vec{\nabla}_{\Theta} \cdot \vec{\nabla}_{\Theta} - \Omega^2 \bar{A}_{\Theta\Theta} \]  \hspace{1cm} (A2)

\[ A_{\Phi\Phi} = \vec{\nabla}_{\Phi} \cdot \vec{\nabla}_{\Phi} - \Omega^2 \bar{A}_{\Phi\Phi} \]  \hspace{1cm} (A3)

\[ A_{\chi\Theta} = \vec{\nabla}_{\chi} \cdot \vec{\nabla}_{\Theta} - \Omega^2 \bar{A}_{\chi\Theta} \]  \hspace{1cm} (A4)

\[ A_{\chi\Phi} = \vec{\nabla}_{\chi} \cdot \vec{\nabla}_{\Phi} - \Omega^2 \bar{A}_{\chi\Phi} \]  \hspace{1cm} (A5)

\[ A_{\Theta\Phi} = \vec{\nabla}_{\Theta} \cdot \vec{\nabla}_{\Phi} - \Omega^2 \bar{A}_{\Theta\Phi} \]  \hspace{1cm} (A6)

\[ B_{\chi} = \vec{\nabla}^2_{\chi} - \Omega^2 \bar{B}_{\chi} \]  \hspace{1cm} (A7)

\[ B_{\Theta} = \vec{\nabla}^2_{\Theta} - \Omega^2 \bar{B}_{\Theta} \]  \hspace{1cm} (A8)

\[ B_{\Phi} = \vec{\nabla}^2_{\Phi} - \Omega^2 \bar{B}_{\Phi} \]  \hspace{1cm} (A9)

Here the gradients, Laplacians and dot products are to be taken treating the \( \bar{X}, \bar{Y}, \bar{Z} \) as Cartesian coordinates, so that, for example,

\[ \vec{\nabla}_{\chi} \cdot \vec{\nabla}_{\Theta} = \frac{\partial \chi}{\partial \bar{X}} \frac{\partial \Theta}{\partial \bar{X}} + \frac{\partial \chi}{\partial \bar{Y}} \frac{\partial \Theta}{\partial \bar{Y}} + \frac{\partial \chi}{\partial \bar{Z}} \frac{\partial \Theta}{\partial \bar{Z}}. \]  \hspace{1cm} (A10)

As in the two-dimensional case, the dot products \( \vec{\nabla}_{\chi} \cdot \vec{\nabla}_{\Theta}, \vec{\nabla}_{\chi} \cdot \vec{\nabla}_{\Phi}, \) and \( \vec{\nabla}_{\Theta} \cdot \vec{\nabla}_{\Phi}, \) vanish since the adapted coordinates are orthogonal (with respect to a Cartesian metric on \( \bar{X}, \bar{Y}, \bar{Z} \)). The other dot products and Laplacians are evaluated with the explicit transformations in Eqs. (6)–(11), from which we find

\[ \vec{\nabla}^2_{\chi} = \frac{a^2 + 2Q}{\chi^3} \]  \hspace{1cm} (A11)

\[ \vec{\nabla}^2_{\Theta} = \frac{\sqrt{Q + a^2 + \chi^2 \cos(2\Theta)}}{\sqrt{Q - a^2 - \chi^2 \cos(2\Theta)}} \frac{(Q - a^2)}{\chi^4} \]  \hspace{1cm} (A12)

\[ \vec{\nabla}^2_{\Phi} = 0 \]  \hspace{1cm} (A13)

\[ \vec{\nabla}_{\chi} \cdot \vec{\nabla}_{\chi} = \frac{Q}{\chi^2} \]  \hspace{1cm} (A14)

\[ \vec{\nabla}_{\Theta} \cdot \vec{\nabla}_{\Theta} = \frac{Q}{\chi^4} \]  \hspace{1cm} (A15)

\[ \vec{\nabla}_{\Phi} \cdot \vec{\nabla}_{\Phi} = 2 \frac{Q + a^2 + \chi^2 \cos(2\Theta)}{\chi^4 \sin^2(2\Theta)} \]  \hspace{1cm} (A16)

where \( Q \) is the function

\[ Q \equiv \sqrt{a^4 + 2a^2 \chi^2 \cos(2\Theta)} + \chi^4. \]  \hspace{1cm} (A17)

The form of the \( \bar{A} \) and \( \bar{B} \) terms in Eqs. (A1)–(A9) follow immediately from the transformation from Eq. (15) to Eq. (16), for example

\[ \bar{A}_{\chi\chi} = Z^2 \left( \frac{\partial \chi}{\partial X} \right)^2 + X^2 \left( \frac{\partial \chi}{\partial Z} \right)^2 - 2XZ \left( \frac{\partial \chi}{\partial X} \right) \left( \frac{\partial \chi}{\partial Z} \right). \]  \hspace{1cm} (A18)

These expressions can then be evaluated, in terms of adapted coordinates, from the explicit coordinate relations in Eqs. (6)–(8), to give

\[ \bar{A}_{\chi\chi} = \frac{a^4 \sin^2(2\Theta) \cos^2 \Phi}{\chi^2}. \]  \hspace{1cm} (A19)
The coefficients needed in the Sommerfeld boundary condition Eq. (18) are

\[ A_{\Theta\Theta} = \frac{\cos^2 \Phi \left[ \chi^2 + a^2 \cos(2\theta) \right]^2}{\chi^4}, \]  

(A20)

\[ A_{\Phi\Phi} = \sin^2 \Phi \frac{Q + a^2 + \chi^2 \cos(2\theta)}{Q - a^2 - \chi^2 \cos(2\theta)} \]  

(A21)

\[ A_{\chi\Theta} = \frac{a^2 \left[ \chi^2 + a^2 \cos(2\theta) \right] \sin(2\theta) \cos^2 \Phi}{\chi^3}, \]  

(A22)

\[ A_{\chi\Phi} = -\frac{a^2 \left[ Q + a^2 + \chi^2 \cos(2\theta) \right] \sin \Phi \cos \Phi}{\chi^3}, \]  

(A23)

\[ A_{\Theta\Phi} = -\frac{\sin(\Phi) \cos(\Phi) \left[ a^2 + \chi^2 \cos(2\Theta) + Q \right] \left[ \chi^2 + a^2 \cos(2\Theta) \right]}{\chi^4 \sin(2\Theta)}. \]  

(A24)

\[ B_{\chi} = \frac{a^2 \left[ \cos^2(\Phi) \left\{ 3a^2 \cos^2(2\Theta) - Q - 2a^2 + \chi^2 \cos(2\Theta) \right\} + Q + a^2 + \chi^2 \cos(2\Theta) \right]}{\chi^3}, \]  

(A25)

\[ B_{\Phi} = \frac{(3Q + a^2 + \chi^2 \cos 2\Theta) \sin(\Phi) \cos(\Phi)}{Q - a^2 - \chi^2 \cos 2\Theta}, \]  

(A26)

\[ B_{\Theta} = \frac{\sqrt{Q + a^2 + \chi^2 \cos(2\Theta)} \left( c \cos^2 \Phi + d \right)}{\chi^6 \sqrt{Q - a^2 - \chi^2 \cos(2\Theta)}} \]  

(A27)

where

\[ c \equiv a^2 \chi^4 \cos(2\Theta) + 2a^4 \chi^4 + 4a^6 \cos(2\Theta) + 4a^4 \chi^2 (\cos(2\Theta))^2 - 4a^4 \cos(2\Theta) - 2a^2 Q \chi^2 - \chi^6 \]  

(A28)

\[ d \equiv \chi^4 \left( a^2 \cos(2\Theta) + \chi^2 \right) \]  

(A29)

The coefficients needed in the Sommerfeld boundary condition Eq. (18) are

\[ \Gamma^{\Theta} = \frac{\chi^2 + a^2 \cos 2\Theta}{\chi^2} \cos \Phi = \cos \Phi \left( 1 + O(a^2/\chi^2) \right) \]  

(A30)

\[ \Gamma^{\Phi} = -\sqrt{\frac{Q + a^2 + \chi^2 \cos(2\Theta)}{Q - a^2 - \chi^2 \cos(2\Theta)}} \sin \Phi = -\cot \Theta \sin \Phi \left( 1 + O(a^2/\chi^2) \right) \]  

(A31)

\[ \Gamma^{\chi} = \frac{1}{\chi^3} \sqrt{(Q + a^2 + \chi^2 \cos(2\Theta))} \left[ Q - a^2 - \chi^2 \cos(2\Theta) \right] = \frac{2 \sin 2\Theta}{\chi} \left( 1 + O(a^2/\chi^2) \right). \]  

(A32)

**APPENDIX B: DETAILS OF THE THREE-DIMENSIONAL SPECTRAL DECOMPOSITION**

The as and \( R_p \)'s in Eq. (25) follow from applying projection with \( \int \int \mathcal{Y}_p \cdot \cdot W \, d\Theta \, d\Phi \),

\[ \alpha_{pk} = \int_0^{2\pi} d\Phi \int_0^\pi d\Theta \, W(\chi, \Theta) \mathcal{Y}_p(\Theta, \Phi) A_{\chi\chi} \mathcal{Y}_k(\Theta, \Phi), \]  

(B1)

and

\[ R_p = \int_0^{2\pi} d\Phi \int_0^\pi d\Theta \, W(\chi, \Theta) \mathcal{Y}_p(\Theta, \Phi) \left( -\sum_{k=0,2,C,S} a_k(\chi) \left[ A_{\Theta\Theta} \frac{\partial^2 \mathcal{Y}_k}{\partial \Theta^2} \right] \right) \]

\[ + \int_0^{2\pi} d\Phi \int_0^\pi d\Theta \, W(\chi, \Theta) \mathcal{Y}_p(\Theta, \Phi) \left( \sum_{k=0,2,C,S} a_k(\chi) \left[ A_{\Theta\Theta} \frac{\partial^2 \mathcal{Y}_k}{\partial \Theta^2} \right] \right) \]
\[ + A_{\Phi \Phi} \frac{\partial^2 Y_k}{\partial \Phi^2} + 2 A_{\Phi \Phi} \frac{\partial^2 Y_k}{\partial \Theta \partial \Phi} + B_{\Phi} \frac{\partial Y_k}{\partial \Phi} + B_{\Phi} \frac{\partial Y_k}{\partial \Phi} \]

\[ + \frac{d a_k}{d \chi} \left[ 2 A_{\chi \Theta} \frac{\partial Y_k}{\partial \Theta} + 2 A_{\Phi \Phi} \frac{\partial Y_k}{\partial \Phi} + B_{\chi} Y_k \right] \] + \left( \chi, \Theta, \sum_k a_k(\chi) Y_k(\Theta, \Phi) \right). \quad \text{(B2)}

For some purposes it is useful to write the \( R_p \) terms as

\[ R_p = \sum_{k=0,2,C,S} G_{pk} a_k + H_{pk} \frac{d a_k}{d \chi} + F \left( \chi, \Theta, \sum_k a_k(\chi) Y_k(\Theta, \Phi) \right) \quad \text{(B3)} \]

in which \( G_{pk} \) and \( H_{pk} \) have the obvious meaning

\[ G_{pk} = - \int_0^{2\pi} d\Phi \int_0^\pi d\Theta \, W(\chi, \Theta) Y_p(\Theta, \Phi) \left[ A_{\Theta \Theta} \frac{\partial^2 Y_k}{\partial \Theta^2} + A_{\Phi \Phi} \frac{\partial^2 Y_k}{\partial \Phi^2} + 2 A_{\Theta \Phi} \frac{\partial^2 Y_k}{\partial \Theta \partial \Phi} + B_{\Theta} \frac{\partial Y_k}{\partial \Theta} + B_{\Phi} \frac{\partial Y_k}{\partial \Phi} \right] \quad \text{(B4)} \]

\[ H_{pk} = - \int_0^{2\pi} d\Phi \int_0^\pi d\Theta \, W(\chi, \Theta) Y_p(\Theta, \Phi) \left[ 2 A_{\chi \Theta} \frac{\partial Y_k}{\partial \Theta} + 2 A_{\Phi \Phi} \frac{\partial Y_k}{\partial \Phi} + B_{\chi} Y_k \right]. \quad \text{(B5)} \]

**APPENDIX C: TWO-DIMENSIONAL LINEARIZED FORMALISM**

If we change our problem to one with line sources that are infinitely long in the \( \bar{z} \) (equivalently \( \bar{Y} \)) direction, then we can make the obvious modifications in the equations of Sec. II to introduce two dimensional adapted coordinates \( \chi \) and \( \Theta \):

\[ \chi = \sqrt{1/r_2} \left\{ \left[ (\bar{x} - a)^2 + \bar{y}^2 \right] \left[ (\bar{x} + a)^2 + \bar{y}^2 \right] \right\}^{1/4} \quad \text{(C1)} \]

\[ \Theta = \frac{1}{2} (\theta_1 + \theta_2) = \frac{1}{2} \tan^{-1} \left( \frac{2 \bar{x} \bar{y}}{\bar{x}^2 - a^2 - \bar{y}^2} \right). \quad \text{(C2)} \]

With these coordinates the 2+1 dimensional version of Eq. (1) takes a form like that of Eq. (16). For convenience we redefine our problem, dividing the original wave equation by \( Q/\chi^2 \), where

\[ Q \equiv \sqrt{a^4 + 2 a^2 \chi^2 \cos(2\Theta) + \chi^4}. \quad \text{(C3)} \]

The result is

\[ \mathcal{L} \Psi \equiv \chi^2 Q^{-1} \left( \nabla^2 - \Omega^2 \partial_\Theta^2 \right) \Psi = A \frac{\partial^2 \Psi}{\partial \chi^2} + B \frac{\partial^2 \Psi}{\partial \Theta^2} + 2 C \frac{\partial^2 \Psi}{\partial \chi \partial \Theta} + D \frac{\partial \Psi}{\partial \chi} + E \frac{\partial \Psi}{\partial \Theta}, \quad \text{(C4)} \]

where

\[ A = 1 - \Omega^2 \frac{a^4 \sin^2(2\Theta)}{Q} \quad \text{(C5)} \]

\[ B = \frac{1}{\chi^2} \left[ 1 - \Omega^2 \left( \frac{a^2 \cos(2\Theta) + \chi^2}{Q} \right)^2 \right] \quad \text{(C6)} \]

\[ C = -\Omega^2 \frac{a^2 \sin(2\Theta) \left( a^2 \cos(2\Theta) + \chi^2 \right)}{Q \chi} \quad \text{(C7)} \]

\[ D = \frac{1}{\chi} \left[ 1 - \Omega^2 \frac{a^2 \left( -a^2 + 3 a^2 \cos(2\Theta) + 2 \chi^2 \cos(2\Theta) \right)}{Q} \right] \quad \text{(C8)} \]

\[ E = +\Omega^2 \frac{2 a^2 \left( 2 a^2 \cos(2\Theta) + \chi^2 \right) \sin(2\Theta)}{Q \chi^2} \quad \text{(C9)} \]
We now expand the standing wave solution $\Psi(\chi, \Theta)$ as

$$\Psi(\chi, \Theta) = \sum_{n=0, 2, 4, \ldots}^N a_n(\chi) \cos n\Theta.$$  \hspace{1cm} (C10)

and our equation becomes

$$\sum_{n=0, 2, 4, \ldots}^N \left[ A \frac{\partial^2 a_n(\chi)}{\partial \chi^2} - n^2 B a_n(\chi) + D \frac{\partial a_n(\chi)}{\partial \chi} \right] \cos n\Theta$$

$$- \left[ 2nc \frac{\partial a_n(\chi)}{\partial \chi} + nEa_n(\chi) \right] \sin n\Theta. \hspace{1cm} (C11)$$

Projecting with $\int_0^{2\pi} \sin m\Theta \cdots d\Theta$, gives zero, by symmetry; projecting with $\int_0^{2\pi} \cos m\Theta \cdots d\Theta$, gives

$$\sum_{n=0, 2, 4, \ldots}^N \alpha_{mn} \frac{\partial^2 a_n(\chi)}{\partial \chi^2} + \beta_{mn} a_n(\chi) + \gamma_{mn} \frac{\partial a_n(\chi)}{\partial \chi} = 0 \hspace{1cm} m = 0, 2, 4, \ldots \hspace{1cm} (C12)$$

where

$$\alpha_{mn} = 4 \int_0^{\pi/2} \cos m\theta \cos n\theta \left[ \frac{\sin^2 (2\theta)}{Q} \right] d\theta \hspace{1cm} (C13)$$

$$\beta_{mn} = -n^2 \int_0^{\pi/2} \cos m\theta \cos n\theta \left[ \frac{(a^2 \cos(2\theta) + \chi^2)^2}{Q} \right] d\theta$$

$$-n \int_0^{\pi/2} \cos m\theta \sin n\theta \left[ \frac{2a^2 (a^2 \cos(2\theta) + \chi^2) \sin(2\theta)}{Q} \right]$$

$$\gamma_{mn} = \frac{1}{\chi} \epsilon_{mn} + \frac{4a^2 \Omega^2}{\pi \chi^2} \left[ - \int_0^{\pi/2} \cos m\theta \cos n\theta \left( \frac{-a^2 + 3a^2 \cos(2\theta) + 2 \chi^2 \cos(2\theta)}{Q} \right) d\theta \right.$$  

$$+ 2n \int_0^{\pi/2} \cos m\theta \sin n\theta \left[ \frac{\sin(2\theta)}{Q} \left( a^2 \cos(2\theta) + \chi^2 \right) \right]$$

where
\[ \epsilon_{mn} \equiv \delta_{m+n,0} + \delta_{m,n} = \begin{cases} 2 & \text{if } m = n = 0 \\ 1 & \text{if } m = n \neq 0 \end{cases} \] (C19)

The integrals needed in Eqs. (C16)–(C18) are all of the form
\[ I = \int_{0}^{\pi/2} \frac{\cos (2P\theta) \sin 2J\theta}{Q(\chi, \theta)} \, d\theta \] (C20)

where \( P \) and \( J \) are integers. With trigonometric identities and with the substitution \( x = \sin \theta \) such integrals can all be expressed in terms of integrals of the form
\[ K_N(k) \equiv \int_{0}^{1} \frac{x^N}{\sqrt{1-k^2x^2}} \sqrt{1-x^2} \, dx . \] (C21)

where
\[ k \equiv \frac{2a\chi}{a^2 + \chi^2} . \] (C22)

For the integrals in Eq. (C20) only even values of \( N \) are needed for \( K_N \). All such integrals can be evaluated in terms of the complete elliptic integral
\[ K(k) \equiv \int_{0}^{1} \frac{1}{\sqrt{1-k^2x^2}} \sqrt{1-x^2} \, dx \quad \text{and} \quad E(k) \equiv \int_{0}^{1} \frac{1}{\sqrt{1-x^2}} \, dx . \] (C23)

To use the elliptic integrals to evaluate the \( K_N(k) \), for even \( N \), we use the relationship (for \( M \geq 0 \))
\[ 0 = \int_{0}^{1} (d/dx) \left[ x^{2M+1} \sqrt{1-k^2x^2} \sqrt{1-x^2} \right] \, dx , \]
\[ = \int_{0}^{1} x^{2M} \left[ (2M + 1) - (2M + 2)(1 + k^2)x^2 + (2M + 3)k^2x^4 \right] \sqrt{1-k^2x^2} \sqrt{1-x^2} \, dx . \] (C24)

This gives us the recursion relation
\[ (2M + 1)K_{2M}(k) - (2M + 2)(1 + k^2)K_{2M+2}(k) + (2M + 3)k^2K_{2M+4}(k) = 0 . \] (C25)

We know that
\[ K_0(k) = K(k) , \]
and we can easily show that
\[ K_2(k) = \frac{1}{k^2} \left[ K(k) - E(k) \right] , \]
so all values of \( K_{2M}(k) \) follow from the recursion relation.

Very efficient computation follows from the results above. At a given value of \( \chi \), only two integrals evaluations must be done, those for \( K(k) \) and \( E(k) \). All values of \( K_2M(k) \) then follow from the recursion relation, and hence all values of \( \alpha_{mn}, \beta_{mn}, \gamma_{mn} \), can be found with negligible computational expense.

[1] several refs on inspiral.