Hello All:

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Mathematical Physics of Quasi-Stationary Inspiral Status Report: 1 November 2004

This month I'm going to focus on the problem of identifying Keplerian orbits in GR. The problem is hard, so I'd like to develop intuition about how to solve it in a simple scalar model. Things aren't even entirely clear in a non-linear scalar model, so the plan will be first to understand the linear case in great detail and in a language which will allow extension to non-linear fields. It is worth noting, in addition, that even though the field equation may be linear in the case I consider here, the coupled matter-field equations of motion are non-linear; solutions do not superpose.

My comments this month amplify and extend my discussion on this topic last month and include a statement of the Kepler law in closed form for relativistic linear scalar field theory.

It is useful at first to consider all possible source distributions and field configurations. We will always be interested only in the case where the fields satisfy the field equations with the given sources. Among these configurations, there are several classes of interest:

1) Time-symmetric solutions consist of a given source and the field associated to it by the half-advanced, half-retarded scalar Green's function.

2) Internally self-consistent solutions (or simply "consistent") solutions have the property that no external force is needed to maintain the specified motion of the sources. That is, consistent solutions solve _both_ the field equations _and_ the matter equations of motion. Obviously, since the field near a given source will diverge, some care will be needed in making this precise.

3) Keplerian solutions are both consistent and time-symmetric. This definition is provisional at the moment. I'd like an alternative method of characterizing these solutions, but it will be delicate to formulate. I strongly suspect the "correct" definition will predict that Keplerian solutions are indeed time-symmetric in the linear theory anyway. The distinction will only matter in the non-linear case.

The first question is how to decide when a given solution is consistent. This brings up the question of the radiation reaction or self-force problem in scalar field theory. There is quite a literature on this, most of which I'll admit I don't understand, but for definiteness I'll fix a convention following Dirac. For a given point source, one calculates the flux of field momentum out of a small tube surrounding its world-line through space-time, and equates this with the change of momentum of the source itself. The results, of course, have singular parts, but Dirac argues that these may be canceled against one another via a sort of renormalization scheme. The end result is a system of equations for the coupled matter-field equations which are free of divergences and which, if this scheme is applied to electromagnetism, agree with experiment. Good enough for me.

In the scalar theory,

\[
\dot{p}_a^i = -q_i \nabla_a \phi_{\text{ext}} - q_i \sum_{j \neq i} \nabla_a \phi_{\text{sym}}^j
\]

Here, the left side is the rate of change of the momentum of the ith point source, \(\phi_{\text{sym}}^j\) is the time-symmetric solution of the field equations associated to the jth point source, and \(\phi_{\text{ext}}\) is the external field determined by boundary conditions on the field.
This may be written as
\[ \phi_{\text{ext}} = \frac{1}{2} (\phi_{\text{in}} + \phi_{\text{out}}) \]

where \( \phi_{\text{in}} \) is the usual "pure radiation" (i.e., homogeneous) field determined by
\[ \phi_{\text{actual}} = \phi_{\text{in}} + \sum_i \phi_{\text{ret}}^i \]

and a similar definition holds for \( \phi_{\text{out}} \), with the retarded solutions used here for each source replaced by the advanced. These define pure radiation fields which agree with the actual field near past and future null infinity, respectively, whence the "in" and "out" nomenclature.

With this matter equation of motion, it is possible to ask whether any solution of the field equations for a given source is or is not consistent. All one needs to do is subtract the time-symmetric solution for a particular given source from the given field configuration, take the gradient of the difference at the source, multiply by minus the charge of that source, and compare the resulting vector with the time rate of change of the momentum of the source itself. (There can be a couple subtleties in this procedure in general, but for binary circular orbits they don't arise.) If these two vectors are equal for each individual point source, the solution is consistent.

What does this mean for binary orbits? Let us specify two point sources with charge \( q \) and mass \( m \) moving at identical constant speeds \( v \) about a common center in a circle of radius \( r \), as viewed from the center-of-mass inertial frame. We focus first on the case where \( \phi_{\text{ext}} \) vanishes, meaning the actual field is the superposition of the time-symmetric fields for each individual source. We can easily calculate the gradient of the time-symmetric field associated to one source, and the rate of change of momentum of the other. The two are equal when
\[ \frac{m v^2}{r} = \frac{q^2}{4r^2} \frac{1 + 2v\sqrt{1 - \sigma^2} + v^2}{(1 + v\sqrt{1 - \sigma^2})^3} \]

where \( \sigma \) is a function of \( v \) associated with the time it takes light to cross the orbit. It is determined implicitly by
\[ \sigma = \cos(v\sigma) \]

This formula is correct (provided its not incorrect, which I'm checking) to all orders and includes all relativistic effects and time-delay effects associated with the field. It imposes a constraint on the four parameters \((q, m, r, v)\) which describe the family of source distributions of interest. It is the Kepler law for relativistic linear scalar field theory. Since this result is so cumbersome, I let Maple pull out the first couple relativistic corrections to the classical result:
\[ \frac{m v^2}{r} = \frac{q^2}{4r^2} \left( 1 - v^2 - 3v^4 + \cdots \right) \]

(The next term is \( 113/9 v^6 \), so the surprising simplicity ends here.) The first correction term is entirely due to time dilation effects. The second brings in the time delay in field propagation.

Now, from the point of view of the GR problem, what we have done here is not quite the right thing. We have begun by looking at all time-symmetric solutions and then picking out of those all which are consistent, giving the Kepler law. In GR, however, inconsistent solutions do not exist. They cannot be constructed, or at least will have naked singularities if they can. The analogue of the GR problem we will ultimately face will involve beginning with a set of all consistent solutions and then identifying those which are Keplerian. I think it will be useful to ask and answer this question already in the scalar theory. To do this, we must figure out how to characterize the complete class of consistent solutions to the scalar field problem and a scheme to identify the Keplerian ones within this class. This should not be difficult.
A non-Keplerian source certainly generates a well-defined time-symmetric field, but requires an external force to maintain its orbit. Rather than supply that force "by hand," we do so using the external radiation field $\phi_{\text{ext}}$. Such an external field can certainly stabilize the orbit and make the solution consistent, but will not be unique. One can always add to any consistent solution of the field equations a homogeneous solution whose gradient vanishes at the two sources, yielding another consistent solution. We all suspect this freedom to add a homogeneous solution may be eliminated by some sort of energy minimization procedure, and that by further minimizing radiative energy under perturbations of the orbital parameters, we may once again recover the Keplerian orbits. That is, non-Keplerian consistent orbits do exist, but require a bit of radiative energy to stabilize. These principles will guide the search for Kepler's law in GR.

To summarize, Keplerian solutions form the intersection of consistent solutions with time symmetric solutions in the space of all histories for the coupled matter-field variables. It is trivial to search among time-symmetric solutions for those which are consistent, but this will not be possible in GR. So, we need to have a technique to search among consistent solutions for those which are time-symmetric. Moreover, we need to have an alternative characterization of the time-symmetric solutions we seek which does not demand linearity of the field equation. This too should be possible and should correspond in some sense to an energy minimization scheme.

The above questions can be investigated analytically in the linear case, but the really interesting application will occur in the non-linear case. In GR, we will be constructing iterative Green's function solutions of the field equations and will need to decide which of them are approximately Keplerian, and to what degree of approximation. I hope the analytic work in the linear case will enable some quantitative analysis of these problems. The linear work should suggest the things to be calculated numerically in the non-linear problem. I'm trying to have some proposals by next month.

Up next: Stephen and Mike this Friday.