FORCES ON CONDUCTORS

We are developing several methods for calculating the electric field, but ultimately we need to find the force produced by the field. In general this is not a trivial problem, but in the case of conductors it is. Consider the surface of an arbitrary conductor carrying a surface charge density $\sigma$:

If we look at an infinitesimal portion of the surface it will be a plane:

We now draw an imaginary box:

Then:

$$\sum \vec{E} \cdot d\vec{A} = \sum \vec{E} \cdot d\vec{A}_{\text{top}} + \sum \vec{E} \cdot d\vec{A}_{\text{sides}} + \sum \vec{E} \cdot d\vec{A}_{\text{bottom}} = E_a + 0 + 0$$

$$= E_a = 4\pi k \sigma a \rightarrow \vec{E} = 4\pi k \sigma \hat{n}$$

The question is: what force does this produce on the surface? This is a subtle question. To answer it we realize that the field lines originate on one charge and end on another. Although we have not yet drawn the other charge, it is there. Because of the planar symmetry we can represent it as another plane of charge:
of opposite sign. Now consider the fields produced by these sheets of charge:

![Diagram of charge sheets]

In regions (1) and (3) they cancel. In region (2) they add. They are of equal magnitude. Hence the field in (2) is half due to the charge on the conductor and half due to the other sheet. But the field due to the charge on the conductor does not exert a force on itself (remember of definition of \( E \)). Thus the force on the conductor is:

\[
F = \frac{1}{2} (4\pi k \sigma) \sigma a
\]

Thus the force/area is:

\[
\frac{F}{A} = 2\pi k \sigma^2
\]

Since the two planes always have opposite sign, the force is always directed outward. Thus the force on the conductor is given by:

\[
\frac{\vec{F}}{A} = 2\pi k \sigma^2 \hat{n}
\]

regardless of the shape of the conductor or the arrangement of charges elsewhere in the universe!

**EXAMPLE OF FORCES ON CONDUCTORS**

Consider two large parallel conducting planes separated by a distance, \( L \), small compared to the dimensions of the planes. One carries a charge \( Q_1 \) and the other a charge \( Q_2 \). We wish to find the force on one plate due to the other.
The first question to answer is how the charges arrange themselves. Since the plates are large compared to their separation we assume that the charges are uniformly distributed on the surface. But the question remains as to what fraction is on the top surface. Not knowing, we split it as shown in the sketch:

\[
\begin{array}{c}
Q_{2o} \\
Q_{2i} \\
Q_{li} \\
Q_{1o}
\end{array}
\]

We then have:

\[
Q_1 = Q_{li} + Q_{1o} \\
Q_2 = Q_{2i} + Q_{2o}
\]

Next we draw an imaginary box as shown:

Since the field is zero inside the conductors and parallel to the sides of the box in the region between the plates, the number of lines leaving the box is zero. Hence there must be no charge inside the box. Thus:

\[
Q_{li} + Q_{2i} = 0
\]

This gives us three equations in four unknowns. To get the fourth equation we note that if we go a long way away the system will look like a point charge. Hence the magnitude of the field above the planes and below them will be the same at large distance. Thus we draw two more boxes:
Since the number of lines leaving each box is the same the charge inside each must also be the same:

\[ Q_{10} = Q_{20} \]

This gives us four equations in four unknowns. Since they are not homogeneous we can solve them. We should note that this last equation is a little shaky since we only know that the field at large distance is the same top and bottom. We have assumed that it is also the same closer to the planes so that the number leaving the sides of the box is the same top and bottom. We will put this on a firmer basis a bit later when we discuss energy.

We can now solve the equations:

\[ Q_1 = Q_{11} + Q_{10} = -Q_{21} + Q_{20} \]

\[ Q_2 = Q_{21} + Q_{20} \]

\[ \therefore Q_1 + Q_2 = 2Q_{20} \rightarrow Q_{20} = \frac{Q_1 + Q_2}{2} \]

\[ Q_{2i} = Q_2 - Q_{20} = \frac{Q_2 - Q_1}{2} \]

We next find the force on plate two in the usual fashion. It consists of the force on the bottom surface (pointed down) plus the force on the top surface (pointed up). Hence the net force up is:

\[ F_{\text{up}} = 2\pi k \left( \frac{Q_{20}}{A} \right)^2 A - 2\pi k \left( \frac{Q_{2i}}{A} \right)^2 A = \frac{2\pi k}{A} \left[ \left( \frac{Q_1 + Q_2}{2} \right)^2 - \left( \frac{Q_2 - Q_1}{2} \right)^2 \right] = \frac{2\pi k Q_1 Q_2}{A} \]