We now consider the effect of fields on dipoles. The situation is as shown in the following drawing.

In general, the field produces both a torque and a force.

**Torque**

\[
\vec{\tau} = \vec{R} \times \vec{F} = \ell \times q \vec{E} \left( \vec{r} + \frac{\ell}{2} \right) + \left( -\frac{\ell}{2} \right) \times (-q) \vec{E} \left( \vec{r} - \frac{\ell}{2} \right) \\
= \frac{\vec{p}}{2} \times \vec{E} \left( \vec{r} + \frac{\ell}{2} \right) + \frac{\vec{p}}{2} \times \vec{E} \left( \vec{r} - \frac{\ell}{2} \right)
\]

Now let \( \ell \to 0 \). Then

\[
\vec{\tau} = \vec{p} \times \vec{E}(\vec{r})
\]

**Force**

\[
\vec{F} = q \vec{E} \left( \frac{\ell}{2} \right) - q \vec{E} \left( \frac{\ell}{2} \right)
\]

Hence if
\[ \vec{E} \left( \vec{r} + \frac{\vec{P}}{2} \right) = \vec{E} \left( \vec{r} - \frac{\vec{P}}{2} \right) \]

Then \( \vec{F} = 0 \). In other words, a dipole feels no net force in a uniform field. In a non-uniform field there will be a force. To find it we consider a special case in which the dipole is parallel to the field.

Then

\[ \vec{E} = \left[ q \vec{E} \left( \vec{r} + \frac{\vec{P}}{2} \right) - q \vec{E} \left( \vec{r} - \frac{\vec{P}}{2} \right) \right] \hat{x} \]

If \( \vec{E} \) is due to another dipole,

\[ \vec{p}_2 = p_2 \hat{x} \]

located at the origin then

\[ \vec{E} = \frac{2kp_2}{r^3} \hat{x} \]

and

\[ \vec{F} = q \frac{2kp_2}{r^3} \left[ \frac{1}{(r + \frac{\ell}{2})^3} - \frac{1}{(r - \frac{\ell}{2})^3} \right] \hat{x} = \frac{2kqp_2}{r^3} \left[ \frac{1}{(1 + \frac{\ell}{2r})^3} - \frac{1}{(1 - \frac{\ell}{2r})^3} \right] \]

Now \((1 + x)^n = 1 + nx + \ldots\)

\[ \therefore \vec{F} = \frac{2kqp_2}{r^3} \left[ \left(1 - \frac{3\ell}{2r}\right) - \left(1 + \frac{3\ell}{2r}\right) \right] \hat{x} = \frac{2kqp_2}{r^3} \left[ -\frac{3\ell}{r} \right] = -\frac{6kp_1p_2}{r^4} \hat{x} \]

Since \( \tau \sim 1/r^3 \) and \( F \sim 1/r^4 \) it is normally torque that dominates.
Potential Energy of Dipole in Electric Field

Consider assembling the dipole from infinity.

$$\text{PE} = \text{PE}_i + \text{work I do to move charges from infinity to their final positions}$$

First move both to the center of the dipole. Since the charges are equal and opposite, this takes no work. Now move them at right angles to the field to a distance $\ell$ apart. This also takes no work because $\vec{F} = q\vec{E}$ is perpendicular to the distance moved. At this point we have

Now rotate by an angle $\phi$ to

This requires work

$$w = -qE\frac{\ell}{2}\sin\phi - (-q)E\left[-\frac{\ell}{2}\sin\phi\right]$$

$$= -q\ell E \sin\phi = -q\ell E \cos\theta$$

$$= -pE \cos\theta = -\vec{p} \cdot \vec{E}$$

$$\therefore \text{PE} = -\vec{p} \cdot \vec{E}$$

Torque on One Dipole Due to Another
\[ \ddot{\mathbf{r}} = \ddot{\mathbf{p}}_2 \times \mathbf{E} = \ddot{\mathbf{p}}_2 \times \frac{k}{r^3} \left[ 3(\mathbf{\hat{p}}_1 \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}} - \mathbf{\hat{p}}_1 \right] = \frac{k}{r^3} \left[ 3(\mathbf{\hat{p}}_1 \cdot \mathbf{\hat{r}}) \ddot{\mathbf{p}}_2 \times \mathbf{\hat{r}} - \mathbf{\hat{p}}_2 \times \ddot{\mathbf{p}}_1 \right] \]

Suppose \( \ddot{\mathbf{p}}_1 = \mathbf{p}_1 \hat{x} \) and \( \ddot{\mathbf{p}}_2 = \mathbf{p}_2 \hat{y} \). Then

\[ \ddot{\mathbf{r}} = \frac{k}{r^3} \left[ 3\mathbf{p}_1 (\mathbf{-p}_2 \hat{z}) - \mathbf{p}_1 \mathbf{p}_2 \hat{z} \right] = -\frac{2kp_1p_2\hat{z}}{r^3} \]

Hence dipole 2 will rotate counter clockwise, as seen from above page, until it lines up with dipole 1.