1. (a) A very long, thin, insulating rod carrying uniform positive linear charge density \( \lambda \) is oriented vertically. A small bead of mass \( m \) carrying positive charge \( q \) is attached to the rod by means of a light thread of length \( L \), as shown in the sketch at the right. When equilibrium is established, the perpendicular distance from the bead to the rod is \( x \).

(i) Use Gauss’s law to find the magnitude and direction of the electric field \( E \) due to the rod at the location of the bead. 

\[
E = \frac{\lambda}{2\pi \varepsilon_0 x} = \frac{2\lambda k_e}{x} \quad \text{(since } k_e = \frac{1}{4\pi \varepsilon_0})
\]

(ii) Assuming the angle \( \theta \) is small enough that the small angle approximation \( \sin \theta \approx \theta \approx \tan \theta \) holds, derive a formula for \( x \) in terms of \( m \), \( q \), \( L \), \( g \), \( k_e \), and \( \lambda \).

\[
(x = \frac{2\lambda k_e q}{mg})
\]
1. (cont'd)

(b) A thin insulating rod of length 2L lies along the x-axis from -L to L, as shown in the sketch at the right, and it carries nonuniform linear charge density \( \lambda = \sigma |x| \), where \( \sigma \) is a positive constant.

(i) What are the SI units of \( \sigma \)? \[2\]

(ii) Set up an integral and calculate (in terms of \( k_e, \sigma, L, \) and \( y \)) the electrostatic potential \( V \) at \( P \), the point on the positive y-axis having coordinates (0, \( y \)). \[12\]

(iii) Use differentiation to find a symbolic expression for the y-component of the electric field (that is, \( E_y \)) at point \( P \). What will the value of \( E_y \) be if \( L = \sqrt{3} y \)? \[6\]

(b) (i) \( \lambda \) is in \( \frac{C}{m} \) and \( x \) is in m, so \( \sigma \) must have units of \( \frac{C}{m^2} \).

(ii) \( \frac{dV}{dr} = \frac{k_e \lambda}{r} = \frac{k_e \sigma x}{\sqrt{x^2 + y^2}} \Rightarrow V = 2 \int_0^L \frac{k_e \sigma x \, dx}{\sqrt{x^2 + y^2}} \) (by symmetry)

Let \( u = x^2 + y^2 \Rightarrow du = 2x \, dx \), \( \sigma \) is constant.

\[ V = \int_{y^2}^{L^2 + y^2} \frac{k_e \sigma}{\sqrt{u}} \, du = k_e \sigma \int_{y^2}^{L^2 + y^2} u^{-\frac{1}{2}} \, du = k_e \sigma \left[ \frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_{y^2}^{L^2 + y^2} = 2k_e \sigma \left[ \frac{(L^2 + y^2)^{\frac{1}{2}} - y}{y^2} \right] \]

(iii) \( E_y = -\frac{dV}{dy} = -2k_e \sigma \left[ \frac{\frac{y}{y^2 + L^2} - \frac{1}{2} (2y) - 1}{2} \right] \)

\[ \Rightarrow E_y = -2k_e \sigma \left[ \frac{\frac{y}{\sqrt{y^2 + L^2}} - 1}{2} \right] \]

If \( L = \sqrt{3} y \), then \( E_y = -2k_e \sigma \left[ \frac{\frac{y}{\sqrt{y^2 + 3y^2}} - 1}{2} \right] = -2k_e \sigma \left[ \frac{y}{\sqrt{4y^2}} - 1 \right] = -2k_e \sigma \left( \frac{\sqrt{3}}{3} \right) = \frac{4k_e \sigma}{3} \),

\[ \Rightarrow E_y = \frac{4k_e \sigma}{3} \]
2. (a) Consider four point charges $q_1$, $q_2$, $q_3$, and $q_4$ that lie in the plane of the page as shown in the sketch at the right. Imagine a three-dimensional closed surface whose cross section in the plane of the page is indicated.

(i) Which of these charges contribute to the net electric flux through the surface? [2] $q_2$ and $q_3$  

(ii) Which of these charges contribute to the resultant electric field at point P? [2] All of them: $q_1$, $q_2$, $q_3$, and $q_4$.

(iii) Are your answers to (i) and (ii) the same or different? Explain why this is so. [4]

Different. Gauss's law says that the total electric flux through a closed surface (which is a global property) depends only on the net charge inside the surface, whereas the resultant electric field at a point (which is a local property) depends on all the charges present.

(iv) If the net charge enclosed by a gaussian surface is zero, does this mean that the electric field is zero at all points on the surface? Justify your answer. [3]

No. A net enclosed charge of zero ensures that the electric flux $\mathbf{\Phi}_E = \oint \mathbf{E} \cdot d\mathbf{A} = 0$ (globally), but it is not necessarily the case that $\mathbf{E} = \mathbf{0}$ at every point on the surface (locally) in order for this to be true.

(v) If the electric field is zero at all points on a gaussian surface, does this mean that the net charge enclosed by the surface is zero? Justify your answer. [3]

Yes. If $\mathbf{E} = \mathbf{0}$ everywhere on the surface, then the flux $\mathbf{\Phi}_E = \oint \mathbf{E} \cdot d\mathbf{A}$ must certainly be zero, which (by Gauss's law) assures us that $q_{\text{in}} = 0$. 


(b) A slab of insulating material of thickness $d$ lies in the $yz$-plane and carries uniform positive volume charge density $\rho$. The slab is infinite in the $y$ and $z$ directions, and $x$ is measured from the centre of the slab, as shown in the sketch at the right.

(i) Use Gauss's law to find the magnitude of the electric field:
- inside the slab as a function of the distance from the $yz$-plane (that is, find $E$ for $|x| \leq d/2$); \[7\]
- outside the slab (that is, find $E$ for $|x| \geq d/2$). \[3\]

(ii) Given that the energy density of the electric field is $\frac{1}{2} \varepsilon_0 E^2$, use integration, together with the answer to the first part of (b) (i), to find an algebraic expression for the total electrostatic energy stored in a portion of the slab of area $A$ (inside the slab only, not outside it). \[10\]
(b) (ii)

Total energy in a portion of slab having cross-sectional area $A$:

$$\int dU = \int u \, dV = 2 \int_0^{\frac{d}{2}} \left( \frac{\rho x}{2 \varepsilon_0} \right) A \, dx$$

both sides of the centre plane

$$\Rightarrow U = \int_0^{\frac{d}{2}} \frac{\rho^2 A}{2 \varepsilon_0} x^3 \, dx = \frac{\rho^2 A x^4}{3 \varepsilon_0} \bigg|_0^{\frac{d}{2}} = \frac{\rho^2 A (\frac{d}{2})^3}{3 \varepsilon_0}$$

$$\Rightarrow U = \frac{\rho^2 A d^3}{24 \varepsilon_0}$$
3. (a) A parallel-plate capacitor has plate area A and plate separation d. Two different dielectrics (having dielectric constants \( \kappa_1 \) and \( \kappa_2 \)) each fill half the space between the plates, as shown in the figure at the right. In terms of A, d, \( \varepsilon_0 \), \( \kappa_1 \), and \( \kappa_2 \), derive a formula for the capacitance. (Hint: Think about what it means to be connected in series or in parallel.) [6]

(b) Three capacitors with vacuum between their plates (having capacitances 2.0 \( \mu \text{F} \), 1.0 \( \mu \text{F} \), and 5.0 \( \mu \text{F} \)) are connected as shown across a battery of emf \( \varepsilon = 12 \text{ V} \).

(i) What is the capacitance of the single capacitor that is equivalent to the combination of the three capacitors? [5]

(ii) What is the charge on the 2.0 \( \mu \text{F} \) capacitor? [5]

(iii) What is the potential difference across the 5.0 \( \mu \text{F} \) capacitor? [4]

(iv) How much electrostatic energy is stored in the system of capacitors? [3]

(v) If the space between the plates of the 1.0 \( \mu \text{F} \) capacitor is filled with a slab of dielectric material having \( \kappa = 3.0 \), how much energy is now stored in the capacitors? [5]

(vi) As the dielectric slab was inserted in part (v), how much work was done, and was it done by the agent exerting a force to overcome electrostatic repulsion in order to insert the slab, or was it done by electrostatic forces attracting the slab into the space between the plates? [4]

(a) The arrangement as shown is clearly equivalent to two separate capacitors in parallel, each having plate area \( \frac{A}{2} \):

\[
\frac{\varepsilon_0 A}{d} \left( \frac{X_1 + X_2}{2} \right)
\]

The equivalent capacitance is then

\[
\frac{X_1 \varepsilon_0 \left( \frac{A}{2} \right)}{d} + \frac{X_2 \varepsilon_0 \left( \frac{A}{2} \right)}{d}
\]

(P.T.O.)
(b) \( \frac{1}{C} = \frac{1}{2\mu F} + \frac{1}{1\mu F + 5\mu F} = \frac{1}{2\mu F} + \frac{1}{6\mu F} = \frac{3}{6\mu F} + \frac{1}{6\mu F} \)

\[ \frac{1}{C} = \frac{4}{6\mu F} \Rightarrow C = \frac{6\mu F}{4} \Rightarrow C = 1.5\mu F \]

(ii) The charge on the 2\( \mu F \) capacitor is the same as on the \( 1\mu F - 5\mu F \) parallel combination, since it is in series with this combination. Also, this charge is the same as on the single equivalent capacitance found in (i). Therefore,

\[ Q = CV = (1.5\mu F)(12V) \Rightarrow Q = 18\mu C \]

(iii) The p.d. across the 2\( \mu F \) capacitor is \( V = \frac{Q}{2\mu F} = \frac{18\mu C}{2\mu F} = 9V \), so the p.d. across the 5\( \mu F \) capacitor (or, for that matter, across the 1\( \mu F \) capacitor) must be \( 12V - 9V = 3V \).

(iv) \( U = \frac{1}{2} CV^2 = \frac{1}{2} (1.5\mu F)(12V)^2 \Rightarrow U = 108\mu J \)

(v) The filled capacitor now has capacitance 3\( \mu F \) instead of 1\( \mu F \), so the new equivalent capacitance \( C' \) of the system is found from

\[ \frac{1}{C'} = \frac{1}{2} + \frac{1}{3+5} = \frac{1}{2} + \frac{1}{8} = \frac{4}{8} + \frac{1}{8} = \frac{5}{8} \]

\[ \Rightarrow C' = \frac{8}{5} \Rightarrow C' = 1.6\mu F \]. Therefore, the new energy is \( U' = \frac{1}{2} C' V^2 = \frac{1}{2} (1.6\mu F)(12V)^2 \Rightarrow U' = 115.2\mu J \).

(P.T.O.)
(vi) Because the battery remains connected as the dielectric is inserted, charge $\Delta Q$ flows from the battery during the insertion process, with $\Delta Q = C'V - CV = (C' - C)V = (1.6 \mu F - 1.5 \mu F)(12 V)$
$\Rightarrow \Delta Q = (0.1 \mu F)(12 V) = 1.2 \mu C$, and the (positive) work done by the battery on this charge is $(\Delta Q)V = (1.2 \mu C)(12 V) = 14.4 \mu J$.
The total work done, however, is $U' - U = 115.2 \mu J - 108 \mu J = 7.2 \mu J$, so we conclude that there must have been work of $-7.2 \mu J$ done on the dielectric as it was inserted. Because that work was negative, we know that electrostatic forces attracted the slab as it was being introduced between the plates.