1. (a) A square loop of wire 3.0 m on a side is perpendicular to a uniform magnetic field of 2.0 T. A 6.0-V light bulb is in series with the loop, as shown in the figure at the right. The magnetic field is reduced steadily to zero over a time interval $\Delta t$.

(i) Find the value of $\Delta t$ such that the bulb will shine at full brightness during this time. [8]

(ii) In which direction will the current flow? [2]

(b) A toroidal coil of square cross section has inner radius $a$ and outer radius $b$. It consists of $N$ turns of wire and carries a time-varying current $I(t) = I_{\text{max}} \sin \omega t$. A single-turn wire loop encircles the toroid, as shown in the figure at the right. In terms of $\mu_0$ and the quantities given, find expressions for:

(i) the self inductance, $L$, of the toroid; [10]

(ii) the mutual inductance, $M$, of the toroid and loop; [4]

(iii) the maximum emf induced in the loop. [2]
1. (cont'd)  

(b) (i) (cont'd)  

\[ \Phi_B = \int d\Phi_B = \frac{\mu_0 N I}{2\pi} \int_a^b \frac{dt}{r} = \frac{\mu_0 N I}{2\pi} (b-a) \ln \left( \frac{b}{a} \right). \]

Then, \( L = \frac{\Phi_B}{I} \Rightarrow \boxed{L = \frac{\mu_0 N (b-a) \ln \left( \frac{b}{a} \right)}{2\pi}}. \)

(ii) \( M = \frac{\Phi_B}{I_{\text{toroid}}} \), but the magnetic flux through the coil is the same as the flux through the toroid. Therefore,

\[ M = L = \frac{\mu_0 N (b-a) \ln \left( \frac{b}{a} \right)}{2\pi}. \]

(iii) \[ |E|_{\text{max}} = \left| -M \frac{dI}{dt} \right|_{\text{max}} = |-M I_{\text{max}} \omega \cos \omega t| = \omega I_{\text{max}} M \]

\[ \Rightarrow E_{\text{max}} = \frac{\omega \mu_0 N I_{\text{max}} (b-a) \ln \left( \frac{b}{a} \right)}{2\pi}. \]
2. A circuit consists of a 12 V battery, a 3.0 Ω resistor, a 6.0 mH inductor, and a 4.0 μF capacitor, as sketched in the figure at the right. Both of the switches $S_1$ and $S_2$ are initially open.

(a) Switch $S_1$ is closed, leaving $S_2$ open. If $I$ is the current through the inductor:

(i) What is the value of $I$ immediately after $S_1$ is closed? [2]

(ii) What is the value of $I$ after $S_1$ has been closed for a long time? [3]

(iii) From the time $S_1$ is closed, how long will it take $I$ to reach half its final value? [8]

(b) After $S_1$ has been closed for a long time, we simultaneously close $S_2$ and open $S_1$, calling the instant at which this is done time $t = 0$.

(i) What is the current through the inductor at $t = 0$, and what is its direction? [2]

(ii) What will be the maximum electric charge on the capacitor, and at what time $t > 0$ will it first occur? [9]

(a) (i) [Blank]

(ii) $\frac{E}{R} = \frac{12V}{3.0\Omega} = 4.0\, A$

(iii) As derived in class: $I = \frac{E}{R}(1 - e^{-t/\tau})$, where $\tau = \frac{1}{R}$. When $I$ is half its final value of $\frac{E}{R}$, we have $\frac{1}{2} = \frac{E}{R}(1 - e^{-t/\tau})$, or

$\frac{1}{2} = 1 - e^{-t/\tau} \Rightarrow e^{-t/\tau} = \frac{1}{2}$. Taking reciprocals of each side:

$e^{t/\tau} = 2 \Rightarrow \frac{t}{\tau} = \ln 2 \Rightarrow t = \tau \ln 2 = \frac{1}{R} \ln 2 = \frac{6.0 \times 10^{-3}}{3.0} \ln 2 \Rightarrow t \approx 1.4 \times 10^{-3}\, s = 1.4\, ms$

(b) (i) $4.0\, A$, downwards in the sketch.

(ii) The angular frequency, $\omega$, of the LC oscillations is given by $\omega = \frac{1}{\sqrt{LC}} = 2\pi f$ \Rightarrow $f = 2\pi \sqrt{\frac{1}{LC}}$ and the period, $T$, of the oscillation is $T = \frac{1}{f} = 2\pi \sqrt{LC}$. (P.T.O.)
\[
t = 2.4 x 10^{-4} s \\
\]

\[
t = \frac{2}{\pi} \frac{4}{2 \sqrt{lC}} = \frac{2}{\pi} \frac{4}{2 \sqrt{lC}} = \frac{2}{\pi} \frac{4}{2 \sqrt{lC}}
\]

That is period \( \Delta t \) will first reach its maximum value. That is \( \Delta t = 0 \), \( I = I_{\text{max}} \), and \( Q = 0 \). One quarter of a period is \( \Delta t = 0 \) and \( I = I_{\text{max}} \), so \( Q_{\text{max}} = I_{\text{max}} lC \), which requires

\[
Q_{\text{max}}^2 = I_{\text{max}} lC (\frac{8.0 \times 10^{-6} (4.0 x 10^{-6})}{4.0})
\]

\[
Q_{\text{max}} = I_{\text{max}} lC, \quad \text{so} \quad \frac{2C}{lI_{\text{max}}} = \frac{2C}{lI_{\text{max}}} = \frac{2C}{lI_{\text{max}}}
\]

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\[
Q_{\text{max}}^2 = I_{\text{max}} lC (\frac{8.0 \times 10^{-6} (4.0 x 10^{-6})}{4.0})
\]

\[
\text{(b) (cont)}}
\]
One period of an alternating voltage is shown in the sketch above. In terms of $V_{\text{max}}$, calculate the rms voltage, $V_{\text{rms}}$. [9]

(b) A ideal step-down transformer on a utility pole operates at $V_1 = 8.50 \text{ kV}$ on the primary side and supplies electric energy to a number of nearby houses at $V_2 = 120 \text{ V}$, both quantities being rms values.

(i) What is the turns ratio $N_1/N_2$ of the transformer? [3]

(ii) The average rate of energy consumption in the houses served by the transformer is 78.0 kW. What are the respective rms currents $I_1$ and $I_2$ in the primary and secondary of the transformer? [6]

(iii) What is the resistive load in the secondary circuit? [3]

(iv) What is the equivalent resistive load in the primary circuit? [3]

\[(a) \quad V_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T [V(t)]^2 \, dt} = \sqrt{\frac{1}{T} \left[ \left( \frac{V_{\text{max}}}{4} \right)^2 + \left( \frac{3V_{\text{max}}}{4} \right)^2 + \left( -\frac{2V_{\text{max}}}{4} \right)^2 + \left( -V_{\text{max}} \right)^2 \right]}
\]

\[= \sqrt{\frac{1}{4} \left( V_{\text{max}} \right)^2 \left[ 1 + \frac{9}{16} + \frac{9}{16} + 1 \right]} = \sqrt{\frac{1}{4} \left( V_{\text{max}} \right)^2 \left[ \frac{16}{16} + \frac{9}{16} + \frac{9}{16} + \frac{16}{16} \right]}
\]

\[= \sqrt{\frac{\left( V_{\text{max}} \right)^2}{4} \left[ \frac{50}{16} \right]} = \sqrt{\frac{50}{64} \left( V_{\text{max}} \right)^2} = \frac{\sqrt{50}}{\sqrt{64}} V_{\text{max}} = \frac{\sqrt{25} \sqrt{2}}{8} V_{\text{max}},
\]

so

\[V_{\text{rms}} = \frac{5\sqrt{2}}{8} V_{\text{max}}.\]
(b) (i) \[
\frac{N_1}{N_2} = \frac{V_1}{V_2} = \frac{8.5 \text{ kV}}{120 \text{ V}} = \frac{8500 \text{ V}}{120 \text{ V}} = \frac{850}{12} = \frac{425}{6} \approx 71.
\]

(ii) Since \( P = IV \) for both the primary and the secondary,
\[
I_1 = \frac{P}{V_1} = \frac{78,000 \text{ W}}{8500 \text{ V}} \Rightarrow I_1 \approx 9.176 \text{ A} \approx 9.2 \text{ A},
\]
\[
I_2 = \frac{P}{V_2} = \frac{78,000 \text{ W}}{120 \text{ V}} \Rightarrow I_2 = 650 \text{ A}.
\]

(iii) \[
R_2 = \frac{V_2}{I_2} = \frac{120 \text{ V}}{650 \text{ A}} \Rightarrow R_2 \approx 0.1846 \Omega \approx 0.18 \Omega.
\]

(iv) \[
R_1 = \frac{V_1}{I_1} = \frac{8500 \text{ V}}{9.176 \text{ A}} \Rightarrow R_1 \approx 926 \Omega.
\]

[Alternatively, use \( R_1 = (\frac{N_1}{N_2})^2 R_2 = (\frac{425}{6})^2 (0.1846 \Omega) \approx 926 \Omega.\]
4. \( c = 3 \times 10^8 \text{ m/s}; \epsilon_0 = 8.85 \times 10^{-12} \text{ N-m}^2/\text{C}^2; \mu_0 = 4\pi \times 10^{-7} \text{ T-m/A} \)

A sinusoidal monochromatic electromagnetic wave in vacuum propagates outwards uniformly in all directions from a point source located at the origin of a right-handed \( xyz \) coordinate system. The frequency of oscillation of the wave is \( 2.0 \times 10^{12} \) Hz, and at a distance of 2.00 m from the source the rms value of the electric field in the wave is \( 1.20 \times 10^3 \) N/m. Find:

(a) the wavelength of the wave; \[2\]

(b) the average intensity of the wave at a distance of 2.00 m from the source; \[4\]

(c) the power output of the source; \[4\]

(d) the average intensity at a distance of 10.0 m from the source; \[4\]

(e) the maximum value of the magnetic field 10.0 m from the source; \[4\]

(f) the force exerted by the wave on a small, perfectly reflecting piece of foil of surface area 3.00 cm\(^2\) oriented perpendicular to the x-axis at the point \( P(10 \text{ m}, 0, 0) \) on the x-axis; \[4\]

(g) the instantaneous direction of the electric field vector \( \mathbf{E} \) at the point \( P \) of part (f), given that the magnetic field vector \( \mathbf{B} \) at \( P \) points in the positive y-direction at that instant. \[4\]

\[(a) \quad c = \frac{f}{\lambda} \quad \Rightarrow \quad \lambda = \frac{c}{f} = \frac{3 \times 10^8 \text{ m}}{2 \times 10^{12} \text{ Hz}} \quad \Rightarrow \quad \lambda = 1.5 \times 10^{-4} \text{ m} = 150 \mu\text{m}. \]

\[(b) \quad I = \frac{(E_{\text{rms}})^2}{2\mu_0 c} = \frac{(\sqrt{2} E_{\text{rms}})^2}{2\mu_0 c} = \frac{(E_{\text{rms}})^2}{\mu_0 c} = \frac{(1.2 \times 10^3)^2}{(4\pi \times 10^{-7})(3 \times 10^8)} \]

\[\Rightarrow \quad I \approx 3.82 \times 10^3 \text{ W/m}^2. \]

\[(c) \quad I = \frac{P}{4\pi R^2} \quad \Rightarrow \quad P = I \cdot (4\pi R^2) = (3.82 \times 10^3 \text{ W/m}^2)(4\pi \times (2\text{ m})^2) \]

\[\Rightarrow \quad P = 1.92 \times 10^5 \text{ W} = 192 \text{ kW}. \]

\[(d) \quad I = \frac{P}{4\pi R^2} = \frac{1.92 \times 10^5 \text{ W}}{4\pi (10 \text{ m})^2} = \frac{153}{\text{W/m}^2}. \]

\[(e) \quad I = \frac{c(B_{\text{max}})^2}{2\mu_0} \quad \Rightarrow \quad (B_{\text{max}})^2 = \frac{2\mu_0 I}{c} = \frac{2(4\pi \times 10^{-7})(153)}{3 \times 10^8} = 1.28 \times 10^{-12} \]

\[\Rightarrow \quad B_{\text{max}} = 1.13 \times 10^{-6} \text{T} = 1.13 \mu\text{T}. \]
(9) We know that $\mathbf{E}$ points in the $+x$ direction.

\[ \mathbf{E} \times \mathbf{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3.1 \times 10^{-6} & 1 \times 10^{-6} & 1 \\ 3 \times 10^8 & 3 \times 10^{-4} & 2 \end{vmatrix} \approx 2 \times 10^{-4} \]