DAMPED HARMONIC OSCILLATOR

We now consider the more realistic case of an oscillator with some friction (air or mechanical). We assume the damping force is proportional to the velocity. Then:

\[ \vec{F}_D = -\alpha \vec{v} \]

We then have an equation of the form:

\[ -kx - \alpha \dot{x} = m \frac{d^2x}{dt^2} \]

\[ \therefore \frac{d^2x}{dt^2} + \frac{\alpha}{m} \frac{dx}{dt} + \frac{k}{m} x = 0 \]

As usual we try a solution of the form:

\[ x = Ae^{st} \]

Then

\[ s^2 Ae^{st} + \frac{\alpha}{m} s Ae^{st} + \frac{k}{m} Ae^{st} = 0 \]

\[ s^2 + \frac{\alpha}{m} s + \frac{k}{m} = 0 \]

\[ \therefore s = \frac{-\alpha \pm \left[ \left( \frac{\alpha}{m} \right)^2 - 4 \frac{k}{m} \right]^{1/2}}{2} = \frac{-\alpha}{2m} \pm \left[ \left( \frac{\alpha}{2m} \right)^2 - \frac{k}{m} \right]^{1/2} \]

Thus the solutions are

\[ x = e^{-\frac{\alpha t}{2m}} \left[ A_1 e^{i\omega t} + A_2 e^{-i\omega t} \right] = e^{-\frac{\alpha t}{2m}} B \cos(\omega t + \phi) \]

where B and \( \phi \) are arbitrary constants, and

\[ \omega = \left[ \left( \frac{\alpha}{2m} \right)^2 - \frac{k}{m} \right]^{1/2} \]
CASE ONE

Consider the case:

\[
\frac{k}{m} > \left(\frac{\alpha}{2m}\right)^2
\]

In this case we simply have an oscillator at a lower frequency which decreases in amplitude with a time constant \((2m/\alpha)\).

CASE TWO

Consider the case:

\[
\frac{k}{m} < \left(\frac{\alpha}{2m}\right)^2
\]

In this case the solution is:

\[
z = e^{-\frac{\alpha t}{2m}} \left[ A e^{\gamma t} + B e^{-\gamma t} \right] = A e^{-\gamma t} + B e^{(\gamma - \gamma) t}
\]

where

\[
\gamma = \left[ \left(\frac{\alpha}{2m}\right)^2 - \frac{k}{m} \right]^{1/2}
\]

Since \((\alpha/2m) > \gamma\) this is simply the sum of two negative exponentials. In other words the system does not oscillate, but rather decays with two time constants.

CASE THREE

Finally consider the case:

\[
\frac{k}{m} = \left(\frac{\alpha}{2m}\right)^2
\]

In this case we have found only one solution:

\[
z = A e^{-\frac{\alpha t}{2m}}
\]
We need the second. There is a general procedure for finding it. If $y_1$ is one solution to the equation:

$$
\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0
$$

Then

$$
y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx
$$

is the second. In our case

$$
y_2 = e^{-\frac{\alpha t}{2m}} \int \frac{e^{-\frac{\alpha dt}{m}}}{e^{\frac{\alpha t}{m}}} dt = e^{-\frac{\alpha t}{m}} \int \frac{e^{-\frac{\alpha t}{m}}}{e^{\frac{\alpha t}{m}}} dt = te^{-\frac{\alpha t}{2m}}
$$

Check it.

$$
\frac{dy}{dt} = e^{-\frac{\alpha t}{2m}} - \frac{\alpha t}{2m} e^{-\frac{\alpha t}{2m}}
$$

$$
\frac{d^2y}{dt^2} = -\frac{\alpha}{2m} e^{-\frac{\alpha t}{2m}} - \frac{\alpha}{2m} e^{-\frac{\alpha t}{2m}} + \left(\frac{\alpha}{2m}\right)^2 te^{-\frac{\alpha t}{2m}}
$$

$$
\therefore \frac{d^2y}{dt^2} + \frac{\alpha}{m} \frac{dy}{dt} + \frac{k}{m} y = -\frac{\alpha}{m} e^{-\frac{\alpha t}{2m}} + \left(\frac{\alpha}{2m}\right)^2 te^{-\frac{\alpha t}{2m}} + \frac{\alpha}{m} e^{-\frac{\alpha t}{2m}} - 2\left(\frac{\alpha}{2m}\right)^2 te^{-\frac{\alpha t}{2m}} + \frac{k}{m} te^{-\frac{\alpha t}{2m}} = e^{-\frac{\alpha t}{2m}} \left[-\left(\frac{\alpha}{2m}\right)^2 + \frac{k}{m}\right] t = 0
$$

Thus in this case the solution is:

$$
z = Ae^{-\frac{\alpha t}{2m}} + Bte^{-\frac{\alpha t}{2m}} = e^{-\frac{\alpha t}{2m}} [A + Bt]
$$

Case one is called an under-damped oscillator. Case two is an over-damped oscillator. Case three is a critically damped oscillator.