We now calculate energy transport in waves. The easiest way to do this is to consider the energy stored in one wavelength and then multiply that by the number of wavelengths going by a point per second.

The energy in a wavelength is the sum of the kinetic and potential energies. Begin with kinetic energy. We consider a one dimensional case with displacement \( y \) and propagating in \( x \). Then:

\[
\frac{dKE}{dx} = \frac{1}{2} \mu \frac{d}{dt} (\frac{\partial y}{\partial t})^2
\]

\[
= \frac{1}{2} \mu \int_0^\lambda \left( \frac{\partial y}{\partial t} \right)^2 dx
\]

Suppose the wave has the form:

\[
y = A \cos (kx - \omega t)
\]

\[
k = \frac{2\pi}{\lambda}
\]

If it doesn’t, we just do a Fourier analysis of the actual form and the present calculation will give the energy in each component. Note that there is a subtlety here – do the energies add? The answer is yes because different Fourier components are orthogonal to each other. We won’t need to concern ourselves with this for now – next semester in Quantum Mechanics!

Then:

\[
\frac{\partial y}{\partial t} = \frac{d}{dt} \omega \sin (kx - \omega t)
\]
\[
\therefore \frac{KE}{\lambda} = \frac{\mu}{2} A^2 \omega^2 \sin^2 (kx - \omega t) dx
\]

Let
\[
kx = \gamma \rightarrow dx = \frac{1}{k} d\gamma
\]

\[
\therefore \frac{KE}{\lambda} = \frac{\mu}{2} A^2 \omega^2 \left( \frac{k\lambda}{k} \right) \sin^2 \gamma \frac{d\gamma}{k} = \frac{\mu}{2} A^2 \omega^2 \left( \frac{2\pi}{k} \right) \sin^2 \gamma \frac{d\gamma}{k} = \frac{\mu \pi A^2}{2k} \omega^2
\]

Next we need the potential energy. This is more subtle. What we need is the work done to bring the amplitude from A to A + dA in the piece dx. But this is:
\[
d\omega = F dy = F dA \cos (kx - \omega t)
\]

Hence
\[
\frac{PE}{\lambda} = \int_0^A \int_0^\lambda F dA \cos (kx - \omega t) dx
\]

Thus we need F (which is the magnitude of the restoring force). This will depend on the system. However we have the general property:
\[
F = m \frac{\partial^2 y}{\partial t^2} = \mu dx A \frac{\partial^2 y}{\partial t^2} = -\mu dx A \omega^2 \cos (kx - \omega t)
\]

Since we need the magnitude of F we find:
\[
\frac{PE}{\lambda} = \int_0^A \int_0^\lambda \mu dx A \omega^2 dA \cos^2 (kx - \omega t) = \frac{\mu A^2 \omega^2}{2k} = \frac{KE}{\lambda}
\]

This is obviously a general result true for any mechanical wave (we did not need to specify the type of wave in the derivation – it only had to obey Newton’s second law).

Thus we get the general result:
\[
\frac{E}{\lambda} = \frac{\mu A^2 \omega^2 \pi}{k}
\]

We then get the power in the wave as:
Power = \frac{\mu A^2 \omega^2 \pi f}{k} = \frac{\mu A^2 4\pi^2 f^3}{k}

But we know

v = f\lambda = f \frac{2\pi}{k} \rightarrow k = \frac{2\pi f}{v}

Hence

Power = \mu A^2 2\pi^2 f^2 v

We can now apply this result to the waves we have so far considered. For the wave on a string:

\[ v = \left( \frac{T}{\mu} \right)^{1/2} \]

\[ \therefore \text{Power} = A^2 2\pi^2 f^2 \left( T/\mu \right)^{1/2} \]

For a sound wave in air:

\[ \text{Power} = \rho a 2\pi^2 f^2 vA^2 \]

where \( a \) is the cross sectional area. We define intensity as (Power/area). Then:

\[ \text{Intensity} = I = 2\pi^2 \rho f^2 vA^2 \]

Since the ear is logarithmic we define a new unit of intensity, the decibel, or db, as:

\[ \text{db} = 10 \log_{10} \frac{I}{I_0} \]

where

\[ I_0 = 10^{-12} \text{watts/m}^2 \]

\( I_0 \) is close to the threshold of hearing. Then:

\[ \text{db} = 10 \log_{10} \left( \frac{2\pi^2 \rho f^2 vA^2}{10^{-12}} \right) = 10 \left[ \log_{10} \left( \frac{2\pi^2 \rho f^2 v}{10^{-12}} \right) + 2 \log_{10} A_y \right] \]
Now the way we have done the problem, $A_y$ is the displacement amplitude, not the pressure amplitude. When we derived the speed of sound we found the relation between the two:

$$dP = -\kappa \frac{\partial v}{\partial x} = \kappa k A \sin(kx - \omega t)$$

Hence the pressure amplitude is $\kappa k$ times the displacement amplitude. Thus:

$$\text{db} = 10 \left[ \log_{10} \left( \frac{2\pi^2 \rho f^2 v}{10^{-12}} \right) + 2 \log_{10} \left( \frac{A_p}{\kappa k} \right) \right]$$

To see the typical magnitude take:

- $P = 1$ kg/m²
- $f = 400$ Hz
- $v = 330$ m/sec
- $\kappa = \rho v^2 = 1 \cdot 330^2$
- $k = 2\pi f/v = 2\pi \cdot 400/330$

Then

$$\text{db} = 10 \left[ \log_{10} \left( \frac{2\pi^2 \times 1 \times 400^2 \times 330}{10^{-12}} \right) + 2 \log_{10} \left( \frac{A_p}{330^2 \times \frac{2\pi \times 400}{330}} \right) \right]$$

$$= 10 \left[ \log_{10} \left( 1.04 \times 10^{21} \right) + 2 \log_{10} (A_p) - 2 \log_{10} \left( 8.29 \times 10^5 \right) \right]$$

$$= 10 \left[ 21.02 - 2 \times 5.91 + 2 \log_{10} A_p \right]$$

$$= 91.8 + 20 \log_{10} A_p$$

Hence a db level of 120 gives

$$\frac{120 - 91.8}{20} = \log_{10} A_p \Rightarrow A_p = 10^{\left( \frac{120-91.8}{20} \right)}$$

$$\therefore A_p = 25.7 \text{ n/m}^2$$

Compared to $A_{atm} = 1 \times 10^5$ n/m². Small, but enough to produce hearing loss! Now suppose the same amplitude were applied to a plate glass window 2 m on a side. Then:

$$F = PA = 4 \cdot 25.3 = 100 \text{ m} \approx 25 \text{ lbs}$$

But suppose db = 160. Then
\[ A_p = 10 \frac{160-91.8}{20} = 2570 \text{ n/m}^2 \]

and

\[ F = 570 \text{ lbs} \rightarrow \text{Breaks!!} \]