FLUIDS

An ideal fluid is one with no viscosity (friction) and whose properties are isotropic. Consider an infinitesimal cube of such a fluid:

The forces on the cube are due to the earth (gravity) and the fluid around it (pressure). The force diagram is then (y direction not shown).

\[ \begin{align*}
\frac{\partial P}{\partial x} &= \frac{\partial P}{\partial y} = 0 \\
\frac{\partial P}{\partial z} &= -\rho g
\end{align*} \]

Hence the pressure at a given depth is the same everywhere in the fluid. Also:

\[ P = P_0 - \rho gh \]
At the top of the fluid \((z = h)\) we have:

\[
P_{\text{atm}} = P_0 - \rho gh
\]

Or:

\[
P_0 = P_{\text{atm}} + \rho gh \quad \quad P = P_{\text{atm}} + \rho g(h - z)
\]

At the bottom this gives:

\[
P(0) = P_{\text{atm}} + \rho gh
\]

as expected.

Now imagine locking the molecules surrounding the box in place while we remove it and replace it with a massless hollow box. Now release the molecules. Since they are now exactly as they were before, the pressure they exert on the box is the same. But we know what that is:

\[
F_z = \rho g(dx dy dz) = \text{weight of fluid in the box}
\]

displaced by the box.

To see how this works consider the following situation. A sphere of radius \(R\) and density \(\rho\) is placed in a fluid of density \(\rho_f\). It is observed that the sphere sinks a distance \(h < 2R\). Find \(\rho\).

Isolate the sphere and make a force diagram for it.
We need $F_B$. But this is the weight of the fluid displaced when the sphere has sunk a distance $h$. To find this we need the volume of the portion of the sphere which is submerged.

$$dV_s = \pi R^2 \sin^2 \theta \, R \sin \theta \, d\theta$$

$$h = R(1 + \cos \theta) \rightarrow \cos \theta = \frac{h}{R} - 1$$

The limits of integration are then

$$\theta_0 = \cos^{-1}\left(\frac{h}{R} - 1\right) \leq \theta \leq \pi$$

Thus

$$V_s = \int_{\theta_0}^{\pi} \pi R^3 \sin^3 \theta \, d\theta = \pi R^3 \int_{\theta_0}^{\pi} \sin \theta (1 - \cos^2 \theta) \, d\theta = \pi R^3 \left[ -\cos \theta + \frac{\cos^3 \theta}{3} \right]_{\theta_0}^{\pi}$$

$$= \frac{\pi R^3}{3} \left[ -3 + 1 - \sin^2 \theta \right]_{\theta_0}^{\pi} = \frac{2\pi R^3}{3} + \frac{\pi R^3}{3} \left[ 2 \cos \theta_0 + \cos \theta_0 \sin^2 \theta_0 \right]$$

$$= \frac{2\pi R^3}{3} \left[ 1 + \cos \theta_0 \left( 1 + \frac{1}{2} \sin^2 \theta_0 \right) \right] = \frac{2\pi R^3}{3} \left[ 1 + \frac{h}{R} - 1 \left( 1 + \frac{1}{2} \left( 1 - \left( \frac{h}{R} - 1 \right)^2 \right) \right) \right]$$

$$\frac{h=2R}{\rightarrow} \frac{2\pi R^3}{3} \left[ 1 + 1 \left( 1 + \frac{1}{2} \left( 1 - 1^2 \right) \right) \right] = \frac{4\pi R^3}{3} \quad \text{ok}$$

$$\frac{h=R}{\rightarrow} \frac{2\pi R^3}{3} \left[ 1 \right] \quad \text{ok}$$

Then

$$F_B = \rho f g \frac{2\pi R^3}{3} \left[ \frac{3}{2} \frac{h}{R} - 1 - \frac{1}{2} \left( \frac{h}{R} - 1 \right)^3 \right]$$

$$\therefore \rho f g \frac{\pi R^3}{3} \left[ \frac{3}{2} \frac{h}{R} - 1 - \left( \frac{h}{R} - 1 \right)^3 \right] = \frac{4\pi}{3} R^3 g \rho$$
HYDRAULIC PRESS

A common application of these ideas is the hydraulic press found in any garage. It consists of two pistons of different cross sectional areas as shown in the figure:

A force $F_1$ is exerted on the left piston producing a pressure:

$$P_L = \frac{F_1}{A_1}$$

But the pressure is the same everywhere in the fluid at the same height. Hence:

$$P_R = P_L = \frac{F_1}{A_1} = \frac{F_2}{A_2} \Rightarrow F_2 = F_1 \frac{A_2}{A_1}$$

and a small force on the left produces a large force on the right. Did we get something for nothing? No, because to raise a weight on the right a distance $h_2$ we had to transfer a volume of fluid

$$V_2 = h_2A_2$$

from left to right. This means we had to depress the left piston by

$$h_1 = \frac{V_2}{A_1} = \frac{h_2A_2}{A_1}$$

Hence we did work:

$$W_1 = h_1F_1 = h_2(A_2/A_1)F_2(A_1/A_2) = W_2$$

Thus we had to put in the same energy as we got out. We simply exerted a smaller force over a greater distance.
Next we turn to fluid dynamics. This is a very complex subject, and we will consider only the simplest form.

TYPES OF FLUID MOTION

We distinguish three types of fluid motion: Streamline (or laminar) flow with or without friction, and turbulent flow. We begin by considering the frictionless case.

LAMINAR FLOW

If the velocities are not too high the flow will be smooth in the sense that particles of the fluid that are close to one another at one point in the motion will remain so:

![Laminar Flow Sketch]

TURBULENT FLOW

The opposite case is shown in the sketch below:

![Turbulent Flow Sketch]

BERNOULLI’S EQUATION

In the case of frictionless, laminar flow we can use conservation of energy to get a simple result. We suppose that the fluid is incompressible. Then:

![Bernoulli’s Equation Diagram]
Consider an infinitesimal chunk of fluid entering at the left and leaving at the right.

\[ \epsilon_i + W = \epsilon_f + Q \]

\[ (KE + PE)_i + P_1 A_1 v_1 dt = (KE + PE)_f + P_2 A_2 v_2 dt \]

\[ \left( \frac{1}{2} \rho v_1^2 A_1 v_1 dt + \rho gh_1 A_1 v_1 dt \right) + P_1 A_1 v_1 dt = \left( \frac{1}{2} \rho v_2^2 A_2 v_2 dt + \rho gh_2 A_2 v_2 dt \right) + P_2 A_2 v_2 dt \]

But since the fluid is incompressible we must have:

\[ A_1 v_1 dt = A_2 v_2 dt \]

Then

\[ \frac{1}{2} \rho v_1^2 + \rho gh_1 + P_1 = \frac{1}{2} \rho v_2^2 + \rho gh_2 + P_2 \]

or

\[ \frac{1}{2} \rho v^2 + \rho gh + P = \text{const} \]

This is Bernoulli’s equation. It has an enormous number of applications including why airplanes fly and baseballs curve!