GYROSCOPES

We consider a simple gyroscope as shown in the sketch below.

The massless axle AB is free to rotate about the frictionless pivot D. The bicycle wheel shown at A has mass $m_2$ and radius $R_2$ with all the mass at the rim. The wheel at B is the same except it has mass $m_1$ and radius $R_1$. Wheel A is spinning clockwise as seen from B with angular speed $\omega_2$. Wheel B is spinning clockwise as seen from A at $\omega_1$.

We choose an inertial coordinate system with origin at D. Then, at the instant shown

$$\vec{\tau} = m_1 g \ell_1 \sin \theta + m_2 g \ell_2 \cos \theta$$

$$\vec{J} = I_1 \ddot{\omega}_1 + I_2 \ddot{\omega}_2 = \left( m_1 R_1^2 \ddot{\omega}_1 \rightarrow \right) + \left( m_2 R_2^2 \ddot{\omega}_2 \leftarrow \right)$$

The basic governing equation is:

$$\vec{\tau} = \frac{d\vec{J}}{dt}$$

There are now four possibilities. The net $J$ may point right or left. The net torque may be in or out of the board. We can sketch these possibilities as seen from above:
The change in J in an infinitesimal time interval, dt, is:

\[ d\vec{J} = \vec{\tau} dt \]

The resulting \( \vec{J} = \vec{J} + \vec{\tau} dt \) is then as shown in the sketch below.

Hence the axis will rotate by an angle, d\( \omega \), in time dt. From the sketches it can go either clockwise or counterclockwise as seen from above. The direction depends on both \( \vec{\tau} \) and \( \vec{J} \), not on either one alone.

Since d\( \theta \) is an infinitesimal we have:

\[ d\theta = \frac{\tau dt}{J} \rightarrow \omega \rho = \frac{d\theta}{dt} = \frac{\tau}{J} \]

This results in a precession period of:

\[ p = \frac{2\pi}{\omega \rho} = \frac{2\pi J}{\tau} \]

It then precesses in the direction found above.

We have simplified the problem by starting the gyro at just the correct speed (\( \omega \rho \)) to keep it horizontal. If it starts too slow it will need to gain speed. Since this will increase its kinetic energy it will dip to do so (loosing potential energy). In general it will overshoot and then have to rise to slow down. This gives the “nutation” motion we saw in class.

**EXAMPLE**

In the sketches above take:

\[ m_1 = 1 \text{ kg} \quad R_1 = .3 \text{ m} \quad l_1 = .6 \text{ m} \]
\[ m_2 = 1.5 \text{ kg} \quad R_2 = .4 \text{ m} \quad l_2 = .5 \text{ m} \]
\[ \omega_2 = 10 \text{ rev/sec clockwise as seen from B} \]
It is observed that the gyro precesses counterclockwise as seen from above with a period of 4 sec. What is the angular velocity of wheel B?

\[ \tau = m_1 g \ell_1 \hat{\omega}_{\text{in}} + m_2 g \ell_2 \hat{\omega}_{\text{out}} = 1 \cdot 9.8 \times 0.6 \hat{\omega}_{\text{in}} + 1.5 \times 9.8 \times 0.5 \hat{\omega}_{\text{out}} = 1.47 \hat{\omega}_{\text{out}} \]

We can now determine which case we have.

We can rule out (2) and (3) because they are clockwise as seen from above. (1) is out because its torque is directed into the board. Hence we have case (4). This means \( \tilde{J} \) is directed to the right. Thus:

\[ J = m_1 R_1^2 \omega \leftarrow m_2 R_2^2 \tilde{\omega}_2 \]

\[ \therefore \tilde{\omega}_2 = \frac{(J \leftarrow) + (m_1 R_1^2 \omega_1 \rightarrow)}{m_2 R_2^2} \]

But

\[ P = \frac{2 \pi J}{\tau} \rightarrow J = \frac{P \tau}{2 \pi} = \frac{4 \cdot 1.47}{2 \pi} \]

\[ \therefore \tilde{\omega}_2 = \frac{\left( \frac{4 \cdot 1.47}{2 \pi} \right) - (1 \times 0.3^2 \times 20 \pi)}{1.5 \times 0.4^2} = -19.7 \text{ rad/sec} \leftarrow = 19.7 \text{ rad/sec} \rightarrow \]

Hence, \( \omega_2 = 19.7 \text{ rad/sec} \) counterclockwise as seen from A.