HARMONIC OSCILLATOR

We now turn to one of the most important special cases – the harmonic oscillator. To understand its importance we consider one-dimensional motion in an arbitrary conservative force field.

Suppose \( x_0 \) is the equilibrium position and write the potential energy as a Taylor series expansion about the equilibrium position:

\[
V(x) = V(x_0) + \frac{dV}{dx}\bigg|_{x_0}(x - x_0) + \frac{1}{2!}\frac{d^2V}{dx^2}\bigg|_{x_0}(x - x_0)^2 + \ldots + \frac{1}{n!}\frac{d^nV}{dx^n}\bigg|_{x_0}(x - x_0)^n + \ldots
\]

Since we can choose the zero of potential energy anywhere we want, we choose it to be zero at the equilibrium position. We know that:

\[
\frac{dV}{dx} \rightarrow -\frac{dV}{dx}
\]

Hence since \( F = 0 \) at the equilibrium position we have

\[
\frac{dV}{dx}\bigg|_{x_0} = 0
\]

Thus

\[
V = \frac{1}{2!}\frac{d^2V}{dx^2}\bigg|_{x_0}(x - x_0)^2 + \ldots
\]

Hence any system whatsoever will have a potential energy of the form:

\[
PE = \frac{1}{2!}\frac{d^2V}{dx^2}\bigg|_{x_0}(x - x_0)^2 + \ldots = \alpha(x - x_0)^2
\]

Provided we do not go too far from the equilibrium position \(([x - x_0]^3 << [x - x_0]^2)\). Now choose the origin at \( x_0 \). Then:

\[
PE = \alpha x^2
\]

\[
F(x) = -2\alpha a \equiv -kx
\]

Thus all conservative systems will obey the equation:
\[
\frac{d^2x}{dt^2} = -\frac{k}{m}x
\]

if not too disturbed from equilibrium.

This equation has an obvious solution:

\[
x = A \cos \left( \sqrt{\frac{k}{m}} t + \phi \right) = A \cos(\omega t + \phi), \quad \omega = \sqrt{\frac{k}{m}}
\]

where A and \(\phi\) are constants to be determined from the initial conditions.

MATH DIVERSION

Consider the linear, homogeneous, \(n\)th order differential equation:

\[
\frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_0 y = 0
\]

where the \(a_i\) are constants. We can always solve this equation by trying a solution of the form:

\[
y = Ae^{sx}
\]

with \(s\) and \(A\) constant. This will work because:

\[
\frac{de^x}{dx} = e^x
\]

We get

\[
Ase^{sx} + a_{n-1}As^{n-1}e^{sx} + \ldots + a_1Ase^{sx} + a_0 Ae^{sx} = 0
\]

\[
\therefore s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0 = 0
\]

This equation has \(n\) roots. Hence the general solution is:

\[
y = A_1e^{s_1x} + \ldots + A_ne^{s_nx}
\]

with the required \(n\) arbitrary constants. (The case where two or more of the \(s_i\) are the same will require additional work to find all the solutions.)
To see how this is related to our case we solve the equation:

\[
\frac{d^2x}{dt^2} = -kx
\]

By this method

\[
x = Ae^{st}
\]

\[
s^2 Ae^{st} + kAe^{st} = 0
\]

\[\therefore s^2 + k = 0 \rightarrow s = \pm i\sqrt{k} = \pm i\omega
\]

Thus

\[
x = A_1e^{i\omega t} + A_2e^{-i\omega t}
\]

But

\[
e^{iz} = 1 + (iz) + \frac{(iz)^2}{2!} + \ldots + \frac{(iz)^n}{n!} + \ldots
\]

\[
= \begin{pmatrix}
1 & \frac{-z^2}{2!} & \frac{z^4}{4!} & \ldots \\
iz & \frac{-iz^3}{3!} & + \ldots
\end{pmatrix} = \cos z + i \sin z
\]

Then

\[
x = A_1(\cos z + i \sin z) + A_2(\cos z - i \sin z) = (A_1 + A_2)\cos z + i(A_1 - A_2)\sin z
\]

Now let

\[
A_1 + A_2 = b_1 \quad \text{and} \quad i(A_1 - A_2) = b_2
\]

Then

\[
x = b_1 \cos(\omega t) + b_2 \sin(\omega t) = \left(\frac{b^2_1 + b^2_2}{2}\right)^{1/2} \left[ \frac{b_1}{\left(b^2_1 + b^2_2\right)^{1/2}} \cos \omega t + \frac{b_2}{\left(b^2_1 + b^2_2\right)^{1/2}} \sin \omega t \right]
\]
\[
\begin{align*}
\therefore \frac{b_2}{\left( b_1^2 + b_2^2 \right)^{1/2}} &= \sin \phi & \frac{b_1}{\left( b_1^2 + b_2^2 \right)^{1/2}} &= \cos \phi \\
\end{align*}
\]

Thus

\[
x = \left( b_1^2 + b_2^2 \right)^{1/2} \left[ \cos \omega t \cos \phi + \sin \omega t \sin \phi \right] = \alpha \cos (\omega t - \phi)
\]

As before