POTENTIAL ENERGY

We define the change in potential energy to be: \( \text{PE}_f - \text{PE}_i = \text{Work I do to move object from } i \text{ to } f \text{ at constant speed.} \) Notice four important things in this definition. First, it only makes sense if the work done is independent of the path taken. Second, only the change in potential energy is defined. Thus I may pick the zero of potential energy anywhere I like. Third, it is the work I do, not the work the force whose potential energy I am finding does. This insures that adding energy to the object increases its potential energy. Fourth, the motion must be at constant speed. This insures that the work done goes into potential energy and not kinetic energy. But what about other sources of energy? For the definition to apply we consider only the force in question. In other words, if friction were present we would imagine it to be zero and calculate the work I would do if it were absent. Although this may be difficult to do experimentally, we can do it in principle – which is all we need for the definition.

Note that for an infinitesimal motion, \( d\mathbf{r} \), we have:

\[
d\text{PE} = \mathbf{F}_M \cdot d\mathbf{r} = -\mathbf{F} \cdot d\mathbf{r} = -F_x dx - F_y dy - F_z dz
\]

\[
\therefore F_x = -\frac{\partial \text{PE}}{\partial x} \quad F_y = -\frac{\partial \text{PE}}{\partial y} \quad F_z = -\frac{\partial \text{PE}}{\partial z}
\]

or

\[
\mathbf{F} = -\nabla \text{PE}
\]

We will often use this in the future.

As an example, consider raising an object of mass \( m \) a height \( h \) near the earth. Then:

![Diagram showing an object of mass m at height h near the earth with forces F_M and mg]

If the object is to move at constant speed the net force on it must be zero. Hence:

\[
\mathbf{F}_M + \mathbf{F} = 0 \rightarrow \mathbf{F}_M = -\mathbf{F} = mg\mathbf{\hat{y}}
\]

then
To define kinetic energy (the energy due to motion) we again use conservation of energy. Suppose the net force on the object is:

$$\mathbf{F}_T = \mathbf{F}_c + \mathbf{F}_{nc}$$

where $\mathbf{F}_c$ is the total conservative force acting on the object and $\mathbf{F}_{nc}$ is the total non-conservative force acting. Let $m$ be the mass of the object. Then:

$$\left(\mathbf{F}_c + \mathbf{F}_{nc}\right) \cdot d\mathbf{r} = m \frac{d\mathbf{v}}{dt} \cdot d\mathbf{r} = m \mathbf{v} \cdot d\mathbf{v}$$

But

$$\mathbf{F}_c \cdot d\mathbf{r} = -dPE$$

$$\mathbf{F}_{nc} \cdot d\mathbf{r} = dQ_0$$

where $dQ$ is the energy going into heat. Then

$$-dPE + dQ = \frac{1}{2} m (\mathbf{v} \cdot d\mathbf{v}) = d \left( \frac{1}{2} mv^2 \right)$$

Let

$$\frac{1}{2} mv^2 = KE = \text{Kinetic Energy}$$

Then

$$d(PE + KE) = dQ$$

Hence if we define kinetic energy as $\frac{1}{2} mv^2$ we find that the change in (potential energy + kinetic energy) is the “heat energy” that goes into the surroundings (surroundings because we defined $dQ$ as $\mathbf{F}_{nc} \cdot d\mathbf{r}$). Here “heat energy” is any energy due to non-conservative forces. Looking more closely at $dQ$ we note that it consists of two parts: $\mathbf{F}_{nc1} \cdot d\mathbf{r}$ where $\mathbf{F}_{nc1} \cdot d\mathbf{r} > 0$, and
\vec{F}_{nc2} \cdot d\vec{r} \text{ where } \vec{F}_{nc2} \cdot d\vec{r} > 0. \vec{F}_{nc1} \text{ will add energy to the mass while } \vec{F}_{nc2} \text{ will remove it. We can summarize the result as: }

\begin{equation}
(PE + KE)_i + W = (PE + KE)_f + Q
\end{equation}

where

\begin{align*}
W &= \int \vec{F}_{nc1} \cdot d\vec{r} \\
Q &= \int \vec{F}_{nc2} \cdot d\vec{r}
\end{align*}

We will find this result extremely useful.

Next we must look more closely at our original tentative definition of energy. When there is only one particle the definition is fine, but when there is more than one we run into problems. Consider the people in a room with equal numbers on each side of a partition. They are running around randomly and bouncing off the walls. In any given time interval roughly equal numbers hit the partition from each side. Hence the partition moves very little and the work done on it is small. Now suppose the people are ordered so that they all move in the same direction. Now many hit the partition from one side and none from the other. Yet each case has the same kinetic energy. Thus our definition of energy as the ability to do work gives a contradiction: “disordered” energy is not as efficient at doing work as is “ordered” energy. We modify our tentative definition as follows.

For one particle it is a stated. For more than one it is the sum of the individual energies:

\begin{equation}
E = \sum E_i = \sum (PE + KE)_i
\end{equation}

Some of you will have wondered if we have missed anything – especially the most famous form of all, the only equation every non-scientist knows:

\begin{equation}
E = mc^2
\end{equation}

The answer is no. According to relativity a particle has a mass given by its total energy (E) divided by c^2 (c = speed of light). In other words the mass is not constant – as we add potential or kinetic energy we increase m. Thus we could replace our definition of energy by:

\begin{equation}
E = \sum m_i c^2
\end{equation}
COLLECTION OF PARTICLES

Consider a collection of N particles. We locate the n\textsuperscript{th} particle relative to an inertial coordinate system by \( \vec{R}_{\text{cm}} \). We then choose an arbitrary point \( \vec{R}_{\text{cm}} \) and write:

\[
\vec{R}_i = \vec{R}_{\text{cm}} + \vec{r}_i
\]

The particle \( m_i \) has a velocity \( \vec{V}_i \) which we write as:

\[
\vec{V}_i = \frac{d}{dt} (\vec{R}_{\text{cm}} + \vec{r}_i) = \vec{V}_{\text{cm}} + \vec{v}
\]

We then have

\[
\vec{F}_i = m_i \left[ \frac{d\vec{V}_{\text{cm}}}{dt} + \frac{d\vec{v}_i}{dt} \right]
\]

\[
\vec{F}_i \cdot d\vec{R}_i = \vec{F}_i \cdot (d\vec{R}_{\text{cm}} + d\vec{r}_i) = m_i \left[ \frac{d\vec{V}_{\text{cm}}}{dt} + \frac{d\vec{v}_i}{dt} \right] \cdot (d\vec{R}_{\text{cm}} + d\vec{r}_i)
\]

\[
= m_i \frac{d\vec{V}_{\text{cm}}}{dt} \cdot d\vec{R}_{\text{cm}} + m_i \frac{d\vec{v}_i}{dt} \cdot d\vec{r}_i + m_i \frac{d\vec{v}_i}{dt} \cdot d\vec{r}_i + m_i \frac{d\vec{V}_{\text{cm}}}{dt} \cdot d\vec{r}_i + m_i \frac{d\vec{v}_i}{dt} \cdot d\vec{r}_i
\]

\[
= m_i \frac{d\vec{V}_{\text{cm}}}{dt} \cdot d\vec{R}_{\text{cm}} + m_i \frac{d\vec{v}_i}{dt} \cdot d\vec{r}_i + m_i \frac{d\vec{v}_i}{dt} \cdot d\vec{r}_i + m_i \frac{d\vec{V}_{\text{cm}}}{dt} \cdot d\vec{r}_i + m_i \frac{d\vec{v}_i}{dt} \cdot d\vec{r}_i
\]

\[
= m_i \frac{d\vec{v}_i}{dt} \cdot d\vec{r}_i + m_i \frac{d\vec{V}_{\text{cm}}}{dt} \cdot d\vec{r}_i + m_i \frac{d\vec{v}_i}{dt} \cdot d\vec{r}_i + m_i \frac{d\vec{V}_{\text{cm}}}{dt} \cdot d\vec{r}_i + m_i \frac{d\vec{v}_i}{dt} \cdot d\vec{r}_i
\]

\[
= \frac{1}{2} m_i d \left( v_{\text{cm}}^2 \right) + d\vec{V}_{\text{cm}} \cdot m_i \vec{v}_i + \vec{V}_{\text{cm}} m_i \cdot d\vec{v}_i + \frac{1}{2} m_i d \left( v_i^2 \right)
\]

Now

\[
\vec{F}_i = \vec{F}_{\text{ext}} + \sum_{j \neq i} \vec{F}_{j \rightarrow i}
\]

where
\[ \vec{F}_{j \to i} \]

is the force of the \( j^{th} \) particle on the \( i^{th} \) particle.

\[
\therefore \vec{F}_{\text{ext}} \cdot (d\vec{R}_{cm} + d\vec{r}) + \sum_{j \neq i} \vec{F}_{j \to i} \cdot (d\vec{R}_{cm} + d\vec{r}) = \frac{1}{2} m_i \dot{v}_{cm}^2 + \frac{1}{2} m_i \dot{v}_i^2 + d\vec{V}_{cm} \cdot m_i \ddot{v}_i + \vec{V}_{cm} \cdot d\vec{v}_i m_i
\]

There is one of these equations for each particle. We now add them all together:

\[
\vec{F}_{\text{ext}} \cdot d\vec{R}_{cm} + \sum_{i} \vec{F}_{\text{ext}} \cdot d\vec{r}_i + \sum_{j \neq i}^{N} \vec{F}_{j \to i} \cdot d\vec{R}_{cm} + \sum_{j \neq i}^{N} \vec{F}_{j \to i} \cdot d\vec{r}_i = \partial V_{cm}^2 + \frac{1}{2} \sum_{i} m_i d\dot{v}_i^2 + d\vec{V}_{cm} \cdot \sum_{i} m_i \dot{v}_i + \vec{V}_{cm} \cdot \sum_{i} m_i d\dot{v}_i
\]

But

\[
d\vec{R}_{cm} \cdot \sum_{j \neq i}^{N} \vec{F}_{j \to i} = 0
\]

(\text{Newton's 3rd Law}). Now choose \( \vec{R}_{cm} \) so that

\[
\sum_{i} m_i \ddot{v}_i = 0
\]

Then

\[
\sum_{i} m_i \dot{v}_i = \sum_{i} m_i \frac{d\vec{r}_i}{dt} = 0
\]

Thus

\[
\vec{F}_{\text{ext}} \cdot d\vec{R}_{cm} + \sum_{i} \vec{F}_{\text{ext}} \cdot d\vec{r}_i + \sum_{j \neq i}^{N} \vec{F}_{j \to i} \cdot d\vec{r}_i = \partial \left( \frac{1}{2} M \dot{V}_{cm}^2 \right) + \frac{1}{2} \sum_{i} m_i \dot{v}_i^2 + \vec{V}_{cm} \cdot \sum_{i} m_i \dot{v}_i
\]

Since \( m_i \) is constant this becomes
\[
\vec{F}_{\text{ext}} \cdot d\vec{R}_{\text{cm}} + \sum_i \vec{F}_{\text{ext}} \cdot d\vec{r}_i + \sum_{j \neq i}^{i,j} \vec{F}_{j \to i} \cdot d\vec{r}_i = \partial \left( \frac{1}{2} M V_{\text{cm}}^2 \right) + \partial \sum_i \frac{1}{2} m_i v_i^2
\]

Now consider each term

\[
\vec{F}_{\text{ext}} \cdot d\vec{R}_{\text{cm}}
\]

Is the work the external force does in moving the “center of mass” (located by \( \vec{R}_{\text{cm}} \))

\[
d\vec{R}_{\text{cm}} \cdot \sum_i \vec{F}_{\text{ext}} \cdot d\vec{r}_i
\]

is the work done by the total force in moving the particles relative to each other.

\[
\sum_{j \neq i}^{i,j} \vec{F}_{j \to i} \cdot d\vec{r}_i
\]

is the work done by internal forces in changing the shape of the object.

\[
\frac{1}{2} M V_{\text{cm}}^2
\]

is the kinetic energy of the center of mass.

\[
\sum_i \frac{1}{2} m_i v_i^2
\]

is the “internal” kinetic energy of the assembly of particles.

Hence we can summarize results so far as:

\[
dW_{\text{ext}} + dW_{\text{int}} = \partial (KE_{\text{cm}}) + \partial (KE_{\text{int}})
\]

Where \( dW_{\text{int}} \) is the work done in changing the shape of the object. We now split \( \vec{F}_{\text{ext}} \) into three parts: \( \vec{F}_c \) (the conservative part of \( \vec{F}_{\text{ext}} \)), \( \vec{F}_{\text{nc1}} \) (the part of the non-conservative force that adds energy to the system), and \( \vec{F}_{\text{nc2}} \) (the part that removes energy from the system). For \( \vec{F}_c \) we use the fact that:

\[
\partial PE = -\vec{F} \cdot d\vec{r}
\]
Now let:

\[ W = \vec{F}_{c1} \cdot d\vec{R}_{cm} \]

\[ Q = \vec{F}_{c2} \cdot d\vec{R}_{cm} \]

to write

\[ \vec{F}_{ext} \cdot d\vec{R}_{cm} = -\partial P E_{cm} \]

Then

\[ -\partial P E_{cm} + W - Q + \partial W_{int} = \partial (K E_{cm}) + \partial (K E_{int}) \]

or

\[ P E_i - P E_f + W = K E_f - K E_i + Q + \partial (K E_{int}) - \partial W_{int} \]

\[ \therefore (P E + K E)_i + W = (P E + K E)_f + Q + dU \]

where \(dU\) is the change in internal energy of the system. For the present we will suppose that the shape of the object is fixed so that \(dU\) is zero. This will be the subject of Thermodynamics which we will come to later in the semester. Hence we can summarize conservation of energy in the form:

\[ M E_i + W_{added} = M E_f + W_{lost} \]

With the mechanical energy, \(M E\) given by

\[ M E = P E + K E \]

We will find this form very useful throughout the year.