COORDINATE SYSTEMS

We have seen that our universe appears to be three dimensional, and hence, needs three numbers to locate a point. However, those numbers can be chosen in a many different ways. Each choice corresponds to a particular coordinate system. We will be concerned with three systems: Cartesian, cylindrical and spherical. In each case we will write vectors in the form

\[ \vec{A} = A \hat{A} \]

where \( \vec{A} \) is the vector, \( A \) is its magnitude and \( \hat{A} \) is a unit vector pointing in the direction of \( \vec{A} \).

In Cartesian coordinates a vector can be written in the form:

\[ \vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \]

where \( A_x \) is the component in the \( \hat{x} \) direction, and \( \hat{x} \) is the unit vector in the x direction. And, similarly for the other two dimensions. This system is particularly simple because the unit vectors are always in the same direction no matter where in space the vector is located.

In cylindrical coordinates the unit vectors are

\[ \rho \hat{\rho} \] in the direction of increasing \( \rho \)
\[ \theta \hat{\theta} \] in the direction of increasing \( \theta \)
\[ \hat{z} \] as usual

Note, however, that the direction of \( \hat{\rho} \) and \( \hat{\theta} \) is not fixed. To see this, consider points in the x,y plane.
This has serious consequences when we take the derivatives of vectors

\[ \ddot{\mathbf{A}} = A_\rho \dot{\rho} + A_\phi \dot{\theta} + A_z \dot{z} \]

\[ \frac{d\ddot{\mathbf{A}}}{dt} = \frac{dA_\rho}{dt} + A_\rho \frac{d\dot{\rho}}{dt} + \frac{dA_\phi}{dt} \dot{\theta} + A_\phi \frac{d\dot{\theta}}{dt} + \frac{dA_z}{dt} \dot{z} + A_z \frac{d\dot{z}}{dt} \]

Hence, we need \( \frac{d\ddot{\rho}}{dt}, \frac{d\ddot{z}}{dt} \) and \( \frac{d\ddot{\theta}}{dt} \). We know \( \frac{d\ddot{z}}{dt} = 0 \). To find the others consider the sketches

Clearly changing either \( z \) or \( \rho \) will not change the direction of any of the unit vectors. But changing \( \theta \) will

Since vectors are defined by direction and magnitude, not location, we can move \( \hat{\rho}_f \) to start where \( \hat{\rho}_i \) does.
Then $d\hat{\rho}$ is as shown. Now since we are looking for a derivative we will make $d\theta \ll 1$. Then the arc length is the same as the chord

$$d\hat{\rho} = 1 \, d\theta$$

since the magnitude of a unit vector is 1. The direction of $d\hat{\rho}$ is $\perp$ to that of $\hat{\rho}_1$ (or $\hat{\rho}_f$ since $d\theta \ll 1$). But this is $\hat{\theta}$. Hence

$$d\hat{\rho} = d\theta \, \hat{\theta}$$

Then

$$\frac{d\hat{\rho}}{dt} = \frac{d\theta}{dt} \, \hat{\theta}$$

Similarly

$$d\hat{\theta} = -d\theta \, \hat{\rho}$$

$$\therefore \frac{d\hat{\theta}}{dt} = -\frac{d\theta}{dt} \, \hat{\rho}$$

Hence

$$\frac{d\hat{A}}{dt} = \frac{dA}{dt} \hat{\rho} + A_p \frac{d\theta}{dt} \hat{\theta} + \frac{dA_b}{dt} \hat{\theta} - A_\theta \frac{d\theta}{dt} \hat{\rho} + \frac{\partial A_z}{\partial z} \hat{z}$$

$$= \hat{\rho} \left[ \frac{dA_p}{dt} - A_\theta \frac{d\theta}{dt} \right] + \hat{\theta} \left[ \frac{dA_b}{dt} + A_p \frac{d\theta}{dt} \right] + \hat{z} \frac{dA_z}{dt}$$
Spherical Coordinates

\[ \vec{A} = A_r \hat{r} + A_\phi \hat{\phi} + A_\theta \hat{\theta} \]

\[
\frac{d\vec{A}}{dt} = \frac{dA_r}{dt} \hat{r} + A_r \frac{d\hat{r}}{dt} + \frac{dA_\phi}{dt} \hat{\phi} + A_\phi \frac{d\hat{\phi}}{dt} + \frac{dA_\theta}{dt} \hat{\theta} + A_\theta \frac{d\hat{\theta}}{dt} + \frac{dA_\psi}{dt} \hat{\psi} + A_\psi \frac{d\hat{\psi}}{dt}
\]

Clearly changing \( r \) does not change the direction of any of the unit vectors. However, changing either \( \theta \) or \( \phi \) does. First consider changing \( \theta \). This will do nothing to \( \hat{\phi} \) (since \( \hat{\phi} \) is \( \perp \) to the \( r,\theta \) plane). Hence, changing \( \theta \) does the same thing to \( \hat{r} \) and \( \hat{\theta} \) as it did to \( \hat{\rho} \) and \( \hat{\phi} \) in cylindrical coordinates.

\[
d\hat{r} = d\theta \hat{\theta} \quad \quad d\hat{\theta} = -d\theta \hat{r}
\]

Changing \( \phi \) is a bit more complicated. Since \( \phi \) is measured in the \( x,y \) plane while \( r \) and \( \theta \) are not, changing \( \phi \) rotates \( \hat{r} \) in a circle or radius \( 1 \times \sin \theta \)

Hence, \( d\hat{r} = \hat{\phi} \sin \theta \, d\phi \)
On the other hand, it rotates \(\hat{\theta}\) in a circle of radius

\[
1 \times \sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta
\]

Hence \(d\theta = \hat{\phi} \cos \theta \, d\phi\)

Now what happens to \(\hat{\phi}\). In the x, y plane we have

Clearly this is just like \(\hat{\theta}\) in cylindrical coordinates. Thus

\[
d\hat{\phi} = -\hat{\rho} \, d\phi
\]

But we don’t want \(\hat{\rho}\). We want \(\hat{\mathbf{r}}\) and \(\hat{\theta}\),

\[
\begin{align*}
\hat{\mathbf{r}} &= \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta} \\
\hat{\theta} &= -\cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\theta}
\end{align*}
\]
From the sketch

\[ \dot{\rho} = -\hat{r} \sin \theta - \hat{\theta} \cos \theta \]

Hence

\[ d\phi = -\hat{r} \sin \theta \, d\phi - \hat{\theta} \cos \theta \, d\phi \]

Putting it together we have

\[
\begin{align*}
\dot{\hat{r}} &= d\theta \hat{\theta} + d\phi \sin \theta \hat{\phi} \rightarrow \frac{d\hat{r}}{dt} = \frac{d\hat{\theta}}{dt} + \frac{d\phi}{dt} \sin \theta \hat{\phi} \\
\dot{\hat{\theta}} &= -d\theta \hat{r} + d\phi \cos \theta \hat{\phi} \rightarrow \frac{d\hat{\theta}}{dt} = -\hat{r} \frac{d\theta}{dt} + \cos \theta \frac{d\phi}{dt} \hat{\phi}
\end{align*}
\]

Then

\[
\begin{align*}
\frac{d\hat{A}}{dt} &= \frac{dA_r}{dr} \hat{r} + A_r [\frac{d\theta}{dt} \hat{\theta} + \sin \theta \frac{d\phi}{dt} \hat{\phi}] + \frac{dA_\phi}{dr} \hat{\theta} + A_\theta [-\hat{r} \frac{d\theta}{dt} + \phi \cos \theta \frac{d\phi}{dt}] \\
&\quad + \frac{dA_\theta}{dr} \hat{\phi} + A_\phi [-\hat{r} \sin \theta \frac{d\phi}{dt} - \hat{\theta} \cos \theta \frac{d\phi}{dt}]
\end{align*}
\]