WAVES

A wave is a disturbance that repeats itself in time and distance. The wavelength is the repeat distance in space. The period is the repeat time. The frequency is the number of repeats per second or (1/period). The wave propagates with some velocity, v. For any wave the relation between velocity, frequency, and wavelength (\(\lambda\)) is the same:

\[ v = f\lambda \]

Waves may have any shape, but all will have an amplitude (the maximum disturbance) and all will propagate in a medium. For some waves the disturbance may be strictly along the direction of propagation. Such waves are said to be “longitudinal”. Others may have the disturbance strictly perpendicular to the direction of propagation. These waves are called transverse.

In order to propagate there must be a restoring force that opposes the disturbance. For this reason liquids and gases don’t propagate transverse waves.

WAVE EQUATION

Consider a wave of arbitrary shape, f(z), where z is measured along the wave. For simplicity we consider a disturbance in only one dimension. Then the wave might look as shown below.

Suppose the wave is moving to the right with velocity v. At time \(t_0\) the point \(z_0\) on the wave is at location \(x_0\) as seen by an observer at rest with respect to the medium.

At time \(\Delta t\) later the point \(z_0\) has moved to \(x_f = x_0 + v\Delta t\). In other words the value of z as seen at \(x_f\) is what it was at \(x_f - v\Delta t\). Hence:

\[ f(x) = f(x - vt) \]

If the wave had been moving to the left, we would get
f(x) = f(x + vt)

Now consider the equation such a curve must satisfy.

\[
\frac{\partial^2 f}{\partial x^2} = \frac{d^2 f}{dz^2} \frac{\partial^2 f}{\partial z^2} - \frac{d^2 f}{dz^2} \frac{d^2 z}{dz^2}
\]

\[
\frac{\partial f}{\partial t} = \frac{d f}{dz} \frac{\partial z}{\partial t} = \pm v \frac{df}{dz}
\]

\[
\frac{\partial^2 f}{\partial t^2} = \pm v^2 \frac{d^2 f}{dz^2} \frac{\partial z}{\partial t} = v^2 \frac{d^2 z}{dz^2}
\]

Hence

\[
\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}
\]

This is the Wave Equation in one dimension. In three dimensions it obviously becomes:

\[
\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}
\]

Where

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla \cdot \nabla
\]

This is one of the most common partial differential equations (PDEs) in physics.

We can now use the wave equation to calculate the speed of propagation of waves.

**WAVE ON A STRING**

Consider a wave propagating on a string under tension, T, and fixed at both ends.
Let the string have \( \frac{\text{mass}}{\text{length}} = \mu \). Let \( y \) be the displacement of the string from a straight line and ignore gravity (since it won’t be a restoring force). Consider a tiny chunk of the string:

Then the \( y \) component of the total force on the piece of string between \( x \) and \( x + dx \) is:

\[
-T \sin[\theta(x)] + T \sin[\theta(x + dx)] = \mu dx \frac{\partial^2 y}{\partial t^2}
\]

But

\[
\tan[\theta(x)] = \frac{y(x + dx) - y(x)}{dx} = \frac{\partial y}{\partial x} \bigg|_x
\]

\[
\tan[\theta(x + dx)] = \frac{y(x + 2dx) - y(x + dx)}{dx} = \frac{\partial y}{\partial x} \bigg|_{x+dx}
\]

Now if \( \theta \) is small

\[
\tan \theta = \sin \theta = \theta
\]

Thus

\[
-T \frac{\partial y}{\partial x} \bigg|_x + T \frac{\partial y}{\partial x} \bigg|_{x+dx} = \mu dx \frac{\partial^2 y}{\partial t^2}
\]
\[
\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T} \frac{\partial^2 y}{\partial t^2}
\]

From the Wave Equation we get:

\[
v = \left( \frac{T}{\mu} \right)^{1/2}
\]

Note that this is true provided the disturbance is small \((\theta << 1)\) – normally the case.

**SOUND WAVES**

Now consider the propagation of a sound wave in air. For simplicity suppose the sound wave is propagating in a long pipe so that it is one dimensional.

![Diagram of sound waves](image)

This time the disturbance is longitudinal and represented by a displacement of a “plane” of atoms (obviously the atoms are not precisely in a plane for a gas, but on average we can think of them that way). Let the pressure of the gas be \(P_0\) and the disturbance in pressure be \(dP(x,t)\). Then consider the gas between \(x\) and \(x + dx\). The force on this chunk of gas in the \(x\) direction is:

\[
P(x)A - P(x + dx)A = [P_0 + dP(x)]A - [P_0 + dP(x + dx)]A
\]

\[
= [dP(x) - dP(x + dx)]A = dm \frac{\partial^2 y}{\partial t^2} \bigg|_x = \rho A dx \frac{\partial^2 y}{\partial t^2} \bigg|_x
\]

where \(\rho\) is the density of the gas. Now suppose the “compressibility of the gas” is \(\kappa\), where:

\[
\kappa = -\frac{1}{V} \frac{\partial V}{\partial P}
\]

Then
\[ \frac{dP(x)}{\kappa V} = -\frac{dV}{\kappa Adx} = -\frac{1}{\kappa A dx} [y(x + dx) - y(x)] A = -\frac{1}{\kappa} \frac{\partial y}{\partial x} \bigg|_x \]

\[ dP(x + dx) = -\frac{1}{\kappa} \frac{\partial y}{\partial x} \bigg|_{x + dx} \]

\[ \therefore \left[ -\frac{1}{\kappa} \frac{\partial y}{\partial x} \bigg|_x + \frac{1}{\kappa} \frac{\partial y}{\partial x} \bigg|_{x + dx} \right] A = \rho Adx \frac{\partial^2 y}{\partial \alpha^2} \]

Thus

\[ \frac{\partial^2 y}{\partial x^2} = \kappa \rho \frac{\partial^2 y}{\partial \alpha^2} \]

And

\[ v = \left( \frac{1}{\kappa \rho} \right)^{1/2} \]

Note that the compressibility is a partial derivative and we will need to specify exactly what variables are being kept constant. We will worry about that in a bit when we do Thermodynamics.