F - G = A \left( \frac{2m\beta c}{\hbar^2 k} + 1 \right) + B \left( \frac{2m\beta c}{\hbar^2 k} - 1 \right)

Let \( B = \frac{m\beta c}{\hbar^2 k} \implies F - G = A(2\beta c + 1) + B(2\beta c - 1) \)

Before proceeding further, let's look at the time dependence of two various waves. We must multiply e^{i(\pm \omega t)}

- For \( F \left( e^{i(kx - \omega t)} \right) \) moving to right, moving to left
- For \( G \left( e^{i(kx + \omega t)} \right) \) on the right side, moving to left
- For \( A \left( e^{i(kx - \omega t)} \right) \) on the left side, moving to right
- For \( B \left( e^{i(kx + \omega t)} \right) \) on the right side, moving to left

Let's assume that we have a scattering experiment in which we are shooting in particles from the left.

Then A represents incident beam
B reflected
C transmitted
G = 0 since there is no further reflected beam beyond the wall.

\[ F = A + B \text{ (since } G = 0) \]

So \( A + B = A(2\beta c + 1) + B(2\beta c - 1) \)

\[ B = A(2\beta c) \]

\[ \frac{B}{-2\beta c + 2} = \frac{A}{1-i\beta} \]

\[ F = A + B = A(1-i\beta + i\beta) = A \frac{1}{1-i\beta} \]

\[ F = \frac{A}{1-i\beta} \]

\[ R = \text{Reflection Coefficient} \]
\[ T = \frac{1}{1 - |A|^2} \]

\[ T = \frac{1}{1 + B^2} \]
We see from this that \( R + T = 1 \) as it should.

Substituting for the well potential \( V(x) \) for \( \beta \)

\[
\beta = \frac{md}{\hbar^2} \quad \beta_E = \frac{m^2d^2 - \hbar^2 q^2}{\hbar^4 k^2} = \frac{m_0^2}{2\hbar^2 E}
\]

\[
R = \frac{\beta^2}{1 + \beta^2} = \frac{1}{1 + \frac{1}{\beta^2}} = \frac{1}{1 + \frac{2\hbar^2 E}{m_0^2}}
\]

\[
T = \frac{1}{1 + \beta^2} = \frac{1}{1 + \frac{m_0^2}{2\hbar^2 E}}
\]

Please note that the existence of a reflected beam, even for \( E > 0 \), clearly differs from classical case. Classically if \( E > 0 \), all the incident beam will be transmitted.

How about the \( \delta \)-bm barrier? \( \beta = +d \delta > 0 \)

We will get the same type of result for \( E > 0 \), but no bound state (for \( E < 0 \)).

This also differs from the classical case. For an infinite barrier, classically there would be no transmission and all would be reflected.

Quantum mechanically, we will always get some transmission for \( 0 < E < \) depth of barrier. This phenomenon is called tunneling and, as we will see for the finite barrier, will depend on the height and width of the barrier.
The Finite Square Well

\[ V(x) = \begin{cases} -V_0, & -a \leq x < 0 \\ 0, & 0 \leq x < a \\ \infty, & x \geq a \end{cases} \]

As before, bound states \( E \leq 0 \)

General solution \( \psi(x) = A e^{-kx} + B e^{kx} \)

To guarantee \( \psi(x) \) normal, \( \psi(-a) = \psi(a) = 0 \)

Similarly \( \psi(x) = \int e^{-kx} \) \( x > a \)

However, in the well we have \( \frac{d^2\psi}{dx^2} + V_0 \psi = E \psi \)

Let \( l = \sqrt{2m(E + V_0)} \)

So \( \psi(x) = C \sin lx + D \cos lx \)

(Note, when we discussed square well, we pointed out that for a symmetric potential well, the wave function will be either even or odd. This is easily proven for a symmetric well (see problem 2, 15).)

Since the well is finite, we can say that \( \frac{d\psi}{dx} \) is continuous everywhere (in contrast to the infinite square well and the infinite potential well, we do not pick even solutions.)

We then have

\[ \psi(x) = \begin{cases} e^{-kx}, & x > a \\ 0, & 0 \leq x \leq a \\ e^{kx}, & x < 0 \end{cases} \]
\( B.C. \quad \psi(x) \) must vanish at \( x = 0 \Rightarrow F \psi (k a) = 0 \)

\[ \frac{d \psi}{dx} \text{ is continuous at } x = 0 \Rightarrow -k F \psi (k a) = 4 m \hbar q \]

Divide 2nd eqn by 1st \( k = \theta \tan \theta \)

Now both sides are functions of energy \( E \); we would like to solve for possible values of \( E \) (these will be the energy eigenvalues corresponding to the bound states). However, this is a transcendental eqn and not easily solved for \( E \).

However, we could plot \( k^2 / \theta \) as a function of \( E \) and also \( \tan \theta \) as a function of \( E \). The values \( k^2 \) for which there is an \( \theta \) corresponding to the bound state (or states). The book suggests a change in notation for convenience.

Let \( \theta = \theta_0 \) and \( \theta_0 = 2 m - 2 \tan \theta_0 \)

But \( k^2 = -2mE \)

\[ k^2 + \theta^2 = 2m \frac{\hbar^2}{\theta^2} = \frac{\theta_0^2}{\theta_0 - 2} \quad \text{or} \quad (k a)^2 + (\theta a)^2 = 2 \theta_0^2 \]

\( (k a)^2 + z^2 = 2 \theta_0 \Rightarrow k a = \sqrt{2 \theta_0 - z^2} \)

Now our eqn becomes \( \theta = 2 \tan \theta \Rightarrow k a = \theta a \quad \text{tan} \theta = 2 \tan \theta \)

So, our original eqn becomes \( 2 \tan z = \sqrt{2 \theta_0 - z^2} \)

\[ z = \frac{\theta_0 + z^2}{2z + 1} \]

We need to solve for \( z \) to get \( E \), but again, this is transcendental in \( \theta \).

Plot \( \tan z = z \) and \( \sqrt{\theta_0 - z^2} = z \) and look for intersection.
Consider the limiting cases:

1. Wide deep well \( \Rightarrow \sqrt{2mV_0} \) is large \( \Rightarrow z_0 \) is large.

In this case, \( E + V_0 \) is just the energy above the bottom of the well.

These then correspond to the result for the infinite square well.

Note these are just half the splitting (corresponding to \( n \) odd) we get a similar set of splitting for \( n \) even (corresponding to \( n \) even) also note, for finite \( V_0 \) get a finite \( \ell \) of bound states.

But as \( V_0 \) gets larger, we get more and more bound states. For infinite well, we get infinite no. of bound states.