Consider $E < E_0$ first.

For $\psi \equiv V_0 \geq 0$:

$$\psi(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ V_0 & \text{if } x > 0 \end{cases}$$

Continuity of $\psi \Rightarrow A + B = 0$.

$$A + B = \frac{1}{2k^2} (A - B) \Rightarrow A \left( 1 + \frac{1}{2k^2} \right) = -B \left( 1 - \frac{1}{2k^2} \right)$$

Reflection coefficient $R = \left| \frac{B}{A} \right| = \frac{1 + \frac{1}{2k^2}}{1 - \frac{1}{2k^2}} = 1$.

So, even though wave function penetrates the step $(F \neq 0)$, all of it is eventually reflected.

Now for the case $E > E_0$.

$$\psi = A e^{ikx} + B e^{-ikx}$$

Continuity $\Rightarrow A + B = 0$.

$$\frac{d\psi}{dx} = \pm k (A - B) \Rightarrow A + B = \frac{k}{i} (A - B) \Rightarrow A = \frac{i}{k} B$$

$$R = \frac{|B|}{A} = \frac{1 - \frac{1}{2k^2}}{1 + \frac{1}{2k^2}}$$

$$R^2 = \frac{2mE}{k^2 - 2m (E - V_0)}$$

$$R = \sqrt{\frac{2mE}{k^2 - 2m (E - V_0)}}$$
Now, what is the probability of finding particle in a range between $q$ and $q + dq$, where $q > 0$:

$$P = \int_{q}^{q+\Delta q} e^{-2Kx} dx$$

$$= \left[ \frac{1}{-2K} e^{-2Kq} \right]_{q}^{q+\Delta q}$$

$$= \frac{1}{-2K} \left( e^{-2Kq} - e^{-2K(q+\Delta q)} \right)$$

If $\Delta q$ is small, $e^{-2K\Delta q} \ll 1$, we can write:

$$\left[ e^{-2Kq} - e^{-2Kq} - e^{-2K\Delta q} \right] = e^{-2Kq} \left( 1 - e^{-2K\Delta q} \right)$$

So:

$$P = \frac{1}{-2K} e^{-2Kq} \left( 1 - e^{-2K\Delta q} \right)$$

From normalization, $P$ is fixed, and $\Delta q$ can be determined.

Please note that, for $E > 0$, we still get reflection, even through classically there would be no reflection.

Also, in the limit as $\Delta q \to 0$, $P \to 0$ as it should if there is no well.

What does $\psi$ look like?

But if we stop, we get the note: it is not immediately clear here.
Chapter 3: Formalism

As we have seen, QM is based on two concepts:

1) Wave function \( \psi \) - State of the system
2) Operators \( \hat{A} \) - Observables

One can use rules of linear algebra, since wave function can be represented as vectors (e.g., time eigenfunction \( \psi_n(t) \)) and the operators acting on them as causing linear transformations (See Appendix A for a review of linear algebra).

A set of \( N \) quantities, e.g., time eigenfunction \( \psi_n \), can be represented as a column vector in \( N \)-dimensional space. Suppose we use notation \( |\alpha> \) to represent a vector by a column matrix having observational components, \( q_1, q_2, \ldots \), we define the inner product

\[
<\alpha|\beta> = \sum_{k=1}^{N} q_k \bar{b}_k
\]

of two vectors \( |\alpha> \) and \( |\beta> \) as the complex (inner) number

\[
<\alpha|\beta> = \begin{pmatrix} q_1 & q_2 & \cdots & q_N \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}
\]

Also, we can often relate two vectors \( |\alpha> \) and \( |\beta> \) by a linear transformation. In this case, we can write

\[
|\beta> = T|\alpha>
\]

which means

\[
\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1N} \\ t_{21} & t_{22} & \cdots & t_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ t_{N1} & t_{N2} & \cdots & t_{NN} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix}
\]
Of course, there may be an infinite no. of functions in which case the vector "space" is infinite-dimensional. When we normalize the wave function \( \phi(x) = 1 \), we reduce the vector space. The set of all square integrable functions, \( f(x) \) such that 
\[
\int |f(x)|^2 \, dx < \infty
\]
is called Hilbert space.

We have given a general definition of the inner product of two vectors. Now, define the inner product of two functions \( f(x) \) and \( g(x) \) as
\[
\langle f | g \rangle = \int_a^b f(x)^* g(x) \, dx.
\]
If both functions are square integrable (i.e., both are in Hilbert space), then this integral will always exist and converge to a finite number.

You can prove this from Schwarz inequality
\[
\left| \int_a^b f(x) g(x) \, dx \right| \leq \sqrt{\int_a^b |f(x)|^2 \, dx} \sqrt{\int_a^b |g(x)|^2 \, dx}
\]
(CF \( a \cdot b = |a| |b| \cos \theta \leq |a| |b| \Rightarrow \|a\| \cdot \|b\| \leq \|a \cdot b\| \)

Note: From def of inner product
\[
\langle f |g \rangle = \int_a^b f(x)^* g(x) \, dx
\]
\[
\langle f | f \rangle = \int_a^b |f(x)|^2 \, dx
\]
Completeness  Def. A set of functions, \( g_n(x) \) is said to be complete if any other function \( f(x) \) can be expressed as a linear combination of them,

\[
f(x) = \sum_{n=0}^{\infty} c_n g_n(x) \Rightarrow c_n = \langle g_n | f \rangle
\]

where \( \langle g_n | f \rangle \) means \( \int g_n^*(x) f(x) dx \)

Observables

For a Hermitian operator \( \hat{Q} \), the expectation value of an observable \( \langle Q \rangle \) can be simply expressed as

\[
\langle Q \rangle = \int \hat{Q}^* \hat{Q} dx = \langle \psi | \hat{Q} | \psi \rangle
\]

Here the operator \( \hat{Q} \) is simply obtained from \( Q \) by simply replacing \( \frac{\partial}{\partial x} \) by \( \frac{i}{\hbar} \frac{\partial}{\partial x} \) (\( \hbar \) is Planck's constant)

Since the results of measurements (as averages of many measurements) must be real, we have that

\[
\langle Q \rangle = \langle Q \rangle^* \quad \text{(in general)}
\]

These operators \( \hat{Q} \), which correspond to the results of real measurements (as observables) have this property

A Def. A Hermitian operator \( \hat{Q} \) is one for which

\[
\langle f | \hat{Q} | f \rangle = \langle \hat{Q} | f \rangle \quad \text{for all } f.
\]

Hermitian operators represent observables.

It is easy to prove a more general condition. That is, if \( f(x) \) and \( g(x) \) are arbitrary functions, then the above condition leads to an (apparently) more general condition

\[
\langle f | \hat{Q} | g \rangle = \langle \hat{Q} | f \rangle \langle f | g \rangle
\]

We will now prove this.

(See Problem 3.3)