Phys 4420  Recitation 3

Ex. 1, Quiz 1, Item 4)
The electric potential and field outside the conductor when a third charge \( q \) is brought near the conductor.
(NOT REQUIRED in Quiz)

Solution: The existence of \( q_a \) and \( q_b \) in the cavities results in the total charge on the outer surface to be \( Q = q_a + q_b \), which cannot be changed by bringing a third charge near the conductor, since the conductor is isolated.

Suppose the location of \( q \) is at \( r = a \). Then we need to introduce the image charge \( q' \) \((-\frac{q R}{a}) \) at point \( r = a' = \frac{R^2}{a} \). This guarantees that the outer surface \( (r = R) \) is of zero potential and has a total induced charge \( q' \). So to maintain its total charge to be \( Q \) and it to be an equipotential surface, we need to put another charge, \( Q' = Q - q' = q_a + q_b + (\frac{q R}{a}) \) at \( r = 0 \).
Problem 4.23 (in Pollack & Stump)

Item b) What is the fraction of the induced charges induced on the hemispherical boss (see Fig. 2).

Solution. It is simpler to compute the induced charges on the planar part of the conducting surface. According to the method of images, there are three images:

Original charge \( q \) at pt \((0, 0, z_0)\)

First image charge \( q' \) at pt \((0, 0, z_0')\)

Second " " \(-q' \) at pt \((0, 0, -z_0)\)

Third " " \(-q \) at pt \((0, 0, -z_0)\)

where \( q' = -q \frac{R}{z_0}, \quad z_0' = \frac{R^2}{z_0} \)

Potential (in cylindrical coord.): \( (z > 0)\)

\[
\Phi(r, z, \varphi) = \frac{1}{4\pi \varepsilon_0} \left\{ \frac{q}{\sqrt{r^2 + (z-z_0)^2}} - \frac{q}{\sqrt{r^2 + (z+z_0)^2}} \right\}
\]

\( \text{indep. of } \varphi \)

Induced charge density on the \( z = 0 \) plane \( (z > R) \) is given by
\[ \sigma(\rho, \varphi) = \varepsilon_0 \kappa \frac{\partial^2 \phi}{\partial \rho^2} \bigg|_{\rho=0} \]

\[ = \frac{-1}{4\pi} \cdot \left( -\frac{1}{2} \right) \left\{ \frac{-2g_{30}}{(\rho^2 + g_{30}^2)^{3/2}} + \frac{-g_{30}'}{(\rho^2 + g_{30}^2)^{3/2}} \right\} \]

\[ = \frac{-1}{4\pi} \left\{ \frac{g_{30}}{(\rho^2 + g_{30}^2)^{3/2}} + \frac{g_{30}'}{(\rho^2 + g_{30}^2)^{3/2}} \right\} \quad \text{(indep. of } \varphi) \]

Integrating over the planar part \((R \leq \rho < \infty, 0 \leq \varphi < 2\pi)\)

\[ \Rightarrow \quad \Omega' = \int_0^R \rho \, d\rho \int_0^{2\pi} \, d\varphi \quad \sigma = \int_0^\infty \rho \, d\rho \left\{ \frac{g_{30}}{(\rho^2 + g_{30}^2)^{3/2}} \right\} \]

\[ = \frac{-g_{30}}{(R^2 + g_{30}^2)^{3/2}} + \frac{g_{30}'}{(R^2 + g_{30}^2)^{3/2}} \quad \left( : \ g_{30}' = -g_{30} \left( \frac{R}{g_{30}} \right)^3 \right) \]

\[ = \frac{-g_{30}}{(R^2 + g_{30}^2)^{3/2}} \left\{ 1 - \frac{(R^2 + g_{30}^2)^{3/2}}{(R^2 + g_{30}^2)^{3/2}} \left( \frac{R}{g_{30}} \right)^3 \right\} \]

\[ = \frac{-g_{30}}{(R^2 + g_{30}^2)^{3/2}} \left\{ 1 - \frac{R^2}{g_{30}^2} \right\} = \frac{-g_{30}^2}{g_{30}^2 \sqrt{R^2 + g_{30}^2}} \]

This is the induced charge on the planar part of the conducting surface. Note that \(\Omega' < 0\) if \(g > 0\).

The total induced charge on the conducting surface should be \(-g\), so the fraction on the hemi-spherical part is

\[ 1 - \frac{\Omega}{g} = 1 - \frac{(g_{30}^2 - R^2)}{g_{30} \sqrt{R^2 + g_{30}^2}} \quad \text{(Q.e.d.)} \]
Ex. 3. The large-distance behavior of the potential in the upper half-space \((W, z > 0)\) in the last problem.

Solution. According to the method of images, the field of the charge-conductor system in the upper half-space can be described by that of the original charge plus (three) image charges. Obviously, the system consisting of these charges has a total charge \(Q = 0\)

and a total dipole \(p_x = 0 = p_y\)

while

\[
p_z = q_3 + q'_3 - q_3' - q(-30) = 2q_3 + 2q_3' = 2q_3(1 - \frac{R^3}{30^3})
\]

\[
\therefore \quad p = 2q_3(1 - \frac{R^3}{30^3}) \hat{k}
\]

At large distances \((r >> 30)\) in \(z > 0\):

\[
\phi(x) \sim \frac{1}{4 \pi \epsilon_0} \frac{p \cdot \hat{r}}{r^2} = \frac{q_3}{2 \pi \epsilon_0} \frac{1 - \frac{R^3}{30^3}}{r^2} \cos \theta
\]