Problem 1

a (Griffiths 1.11(b))

\[ f(x, y, z) = x^2y^3z^4 \]

\[ \vec{\nabla}f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z} = 2xy^3z^4\hat{x} + 3x^2y^2z^4\hat{y} + 4x^2y^3z^3\hat{z} \]

b (Griffiths 1.15(c))

\[ \vec{v}(x, y, z) = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z} \]

\[ \vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 2x + 2y \]

c (Griffiths 1.18(c))

\[ \vec{v}(x, y, z) = y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z} \]

\[ \vec{\nabla} \times \vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z} = (2z - 2z)\hat{x} + (0 - 0)\hat{y} + (2y - 2y)\hat{z} = 0. \]

Note: You may have recognized that \( \vec{v}(x, y, z) = \vec{\nabla}f \), where \( f = xy^2 + yz^2 \). Thus, by Griffiths’ equation 1.44, \( \vec{\nabla} \times \vec{v} = 0. \)

Problem 2 (Griffiths 1.12)

\[ h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12) \]
The stationary points of \( h \) are found where \( \vec{\nabla} h = 0 \). Thus, to find the peak we set

\[
-60x + 20y - 180 \hat{x} + (-80y + 20x + 280) \hat{y} = 0,
\]

which, of course, can only be true if

\[-3x + y - 9 = 0\]

and

\[x - 4y + 14 = 0.\]

Solving this system of simultaneous equations gives only one solution at the point \((-2, 3)\); i.e., 3 miles north and 2 miles west of South Hadley. This position must correspond to the top of the hill. We know it is a maximum (rather than another type of stationary point) because of the form of \( h(x, y) \) (it’s quadratic, and the coefficients of \( x^2 \) and \( y^2 \) in \( h(x, y) \) are both negative).

\[h(-2, 3) = 10(2(-2)(3) - 3(-2)^2 - 4(3)^2 - 18(-2) + 28(3) + 12 = 720 ft\]

\[\vec{\nabla} h(1, 1) = (-60(1) + 20(1) - 180) \hat{x} + (-80(1) + 20(1) + 280) \hat{y} = -220 \hat{x} + 220 \hat{y}\]

The slope of the hill is the magnitude of the gradient, which is \( 220 \sqrt{2} \text{ ft.} / \text{mile} \). It points uphill in the direction of the gradient, which is to the north-west.

**Problem 3** (Griffiths 1.21)

\( A \) We should obey the parentheses and compute the product they contain first. However, \( \vec{\nabla} \) is an operator, so we should leave it open to act on the right as it is written. Thus,

\[
(\vec{A} \cdot \vec{\nabla}) \vec{B} = (A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z}) \vec{B}
\]

\[
= (A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z}) \hat{x} + (A_x \frac{\partial B_y}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_y}{\partial z}) \hat{y} + (A_x \frac{\partial B_z}{\partial x} + A_y \frac{\partial B_z}{\partial y} + A_z \frac{\partial B_z}{\partial z}) \hat{z}.
\]
\[ \hat{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}} \]

Since the functional form of \( \hat{r}_i \) is identical for \( i \in \{x, y, z\} \), we need only compute one component of \((\hat{r} \cdot \vec{\nabla})\hat{r}\) (the others will be the same with the variables simply interchanged).

\[ ((\hat{r} \cdot \vec{\nabla})\hat{r})_x = \hat{r}_x \frac{\partial \hat{r}_x}{\partial x} + \hat{r}_y \frac{\partial \hat{r}_x}{\partial y} + \hat{r}_z \frac{\partial \hat{r}_x}{\partial z} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left( x \left( \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}} \right) + y \left( \frac{-xy}{(x^2 + y^2 + z^2)^{3/2}} \right) + z \left( \frac{-xz}{(x^2 + y^2 + z^2)^{3/2}} \right) \right) \]

\[ = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \left( x(y^2 + z^2) - xy^2 - xz^2 \right) \]

\[ = 0 \]

Thus, \((\hat{r} \cdot \vec{\nabla})\hat{r} = 0\).

**Problem 4** (Griffiths 1.24(a) - optional)

\[ \vec{A} = x\hat{x} + 2y\hat{y} + 3z\hat{z} \]

\[ \vec{B} = 3y\hat{x} - 2x\hat{y} \]

We’d like to prove product rule (iv) from the text, namely

\[ \vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \]

First, let’s expand the left hand side:

\[ \vec{A} \times \vec{B} = (0 - (-6xz))\hat{x} + (9yz - 0)\hat{y} + (-2x^2 - 6y^2)\hat{z} \]

So, taking the divergence gives

\[ \vec{\nabla} \cdot (\vec{A} \times \vec{B}) = 6z + 9z = 15z \]

Now for the right hand side. First we’ll compute the curls:

\[ \vec{\nabla} \times \vec{A} = (0 - 0)\hat{x} + (0 - 0)\hat{y} + (0 - 0)\hat{z} = 0 \]

\[ \vec{\nabla} \times \vec{B} = (0 - 0)\hat{x} + (0 - 0)\hat{y} + (-2 - 3)\hat{z} = -5\hat{z} \]

So the right hand side is

\[ \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) = 0 - 3z(-5) = 15z \]

**Problem 5** (Griffiths 1.25)
b

\[ T(x, y, z) = \sin x \sin y \sin z \]

\[ \nabla^2 T = -\sin x \sin y \sin z - \sin x \sin y \sin z - \sin x \sin y \sin z \]
\[ = -3T(x, y, z) \]

d

\[ \vec{v}(x, y, z) = x^2 \hat{x} + 3xz^2 \hat{y} - 2xz \hat{z} \]

\[ \nabla^2 \vec{v} = (\nabla^2 v_x)\hat{x} + (\nabla^2 v_y)\hat{y} + (\nabla^2 v_z)\hat{z} \]
\[ = 2\hat{x} + 6x\hat{y} \]

Problem 6 (Griffiths 1.31)

\[ T(x, y, z) = x^2 + 4xy + 2yz^3 \]

\[ \vec{v} = (2x + 4y)\hat{x} + (4x + 2z^3)\hat{y} + (6yz^2)\hat{z} \]

We’d like to prove that

\[ \int_a^b (\nabla^2 T) \cdot d\vec{l} = T(b) - T(a) \]

regardless of the path we choose. First, let’s find the right hand side so we know what to look for on the left.

\[ T(b) - T(a) = T(1, 1, 1) - T(0, 0, 0) \]
\[ = (1)^2 + 4(1)(1) + 2(1)(1)^3 = 7 \]

a

\[ (0, 0, 0) \rightarrow (1, 0, 0) \rightarrow (1, 1, 0) \rightarrow (1, 1, 1) \]

Along this path we have three separate line integrals.

\[ \int_{(0,0,0)}^{(1,1,1)} (\nabla^2 T) \cdot d\vec{l} = \int_0^1 \nabla^2 T \cdot dx \hat{x} \bigg|_{y=0=z} + \int_0^1 \nabla^2 T \cdot dy \hat{y} \bigg|_{x=1,z=0} + \int_0^1 \nabla^2 T \cdot dz \hat{z} \bigg|_{x=1=y} \]
\[ = \int_0^1 2x dx + \int_0^1 4dy + \int_0^1 6z^2 dz \]
\[ = 1 + 4 + 2 = 7 \]
\((0, 0, 0) \rightarrow (1, 1, 1); z = y^2, y = x\)

Now we have three integrals for a different reason (sort of). There’s just one smooth path, but our path integral gets split into components. In each integral we substitute expressions for each variable in terms of the variable of integration.

\[
\int_{(0,0,0)}^{(1,1,1)} (\vec{\nabla} T) \cdot d\vec{l} = \int_0^1 (2x + 4y)dx + \int_0^1 (4x + 2z^3)dy + \int_0^1 6yz^2dz
\]

\[
= \int_0^1 6xdx + \int_0^1 (4y + 2y^6)dy + \int_0^1 6z^2dz
\]

\[
= 3 + \left(2 + \frac{2}{7}\right) + 12\frac{7}{7} = 7
\]

**Problem 7** (Griffiths 1.33)

\[\vec{v}(x, y, z) = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}\]

Stokes’ Theorem says that

\[
\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{A} = \int_P \vec{v} \cdot d\vec{l}.
\]

The right hand side of this equation is easy. Following the arrows in Figure 1.34, and starting from the origin, we have three line integrals:

\[
\int_P \vec{v} \cdot d\vec{l} = \int_0^2 (xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}) \cdot dy\hat{y} + \int_0^2 (xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}) \cdot (dy\hat{y} + dz\hat{z}) + \int_0^0 (xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}) \cdot (-dz\hat{z}).
\]

The second line integral must be split into two integrals: one in the \(y\)-direction and one in the \(z\)-direction. Also, in each integral we should replace the non-integrated variables with appropriate values. In the first path we hold \(x = z = 0\), so the integral vanishes. In the third path we have \(x = y = 0\), so it also vanishes. However, in the second path we have \(x = 0\) and \(z = 2 - y\). Thus,

\[
\int_P \vec{v} \cdot d\vec{l} = \int_0^2 2yzdy + \int_0^0 2y(2 - y)dy + \int_0^2 3zxdz + \int_0^0 3zx(-dz)
\]

\[
= \int_0^2 (2y^2 - 4y)dy
\]

\[
= \frac{16}{3} - 8 = \frac{8}{3}.
\]
For the left hand side of Stokes’ Theorem we note that the positive $x$ direction is the direction of $d\vec{A}$. This is given by the right hand rule according to the path we took above. Thus, we need to find

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{A} = \int \int_S (\vec{\nabla} \times \vec{v}) \cdot (dydz\hat{x}),$$

where

$$\vec{\nabla} \times \vec{v} = (0 - 2y)\hat{x} + (0 - 3z)\hat{y} + (0 - x)\hat{z}.$$ 

There are a few ways to proceed with the double integral from here. Let’s integrate over $z$ first, since $v_x$ doesn’t depend on $z$. The $z$ boundary, however, depends on $y$. Thus,

$$\int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{A} = \int_0^2 dy \int_0^{2-y} dz (0 - 2y\hat{x} + 3z\hat{y} - x\hat{z}) \cdot \hat{x}$$

$$= \int_0^2 dy (2 - y)(-2y)$$

$$= -8/3.$$