Problem 1  (Griffiths 1.35)

a  From product rule (v) we know

\[ f(\vec{\nabla} \times \vec{A}) = \vec{A} \times (\vec{\nabla} f) + \vec{\nabla} \times (f \vec{A}). \]

Integrating both sides over any surface (closed or not) gives

\[ \int_S f(\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \int_S (\vec{A} \times (\vec{\nabla} f)) \cdot d\vec{a} + \int_S (\vec{\nabla} \times (f \vec{A})) \cdot d\vec{a}. \]

The second integral on the right may be changed according to the fundamental theorem for curls, Stokes’ Theorem. Thus,

\[ \int_S f(\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \int_S (\vec{A} \times (\vec{\nabla} f)) \cdot d\vec{a} + \oint_P (f \vec{A}) \cdot d\vec{l}. \]

b  From product rule (iv) we know

\[ \vec{B} \cdot (\vec{\nabla} \times \vec{A}) = \vec{A} \cdot (\vec{\nabla} \times \vec{B}) + \vec{\nabla} \cdot (\vec{A} \times \vec{B}). \]

We just need to integrate both sides over any volume and transform the second integral on the right using the fundamental theorem for divergence, Gauss’ Law, to obtained the desired result:

\[ \int_V \vec{B} \cdot (\vec{\nabla} \times \vec{A}) dV = \int_V \vec{A} \cdot (\vec{\nabla} \times \vec{B}) dV + \oint_S (\vec{A} \times \vec{B}) \cdot d\vec{a}. \]

Problem 2  (Griffiths 1.39)

\[ \vec{v} = (r \cos \theta) \hat{r} + (r \sin \theta) \hat{\theta} + (r \sin \theta \cos \phi) \hat{\phi} \]

\[ \vec{\nabla} \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (v_\phi) \]

\[ = \frac{1}{r^2} \frac{\partial}{\partial r} (r^3 \cos \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin^2 \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (r \sin \theta \cos \phi) \]

\[ = 3 \cos \theta + 2 \cos \theta - \sin \phi = 5 \cos \theta - \sin \phi \]
We’d like to verify the divergence law
\[ \int_V \nabla \cdot \vec{v} dV = \oint_S \vec{v} \cdot d\hat{a} \]
for a hemisphere of radius \( R \) whose base is centered at the origin and lies in the xy-plane. For the left hand side we integrate over all three variables using the volume element \( dV = r^2 \sin \theta dr d\theta d\phi \).

\[
\int_V \nabla \cdot \vec{v} dV = \int_0^R dr \int_0^\frac{\pi}{2} d\theta \int_0^{2\pi} d\phi (5 \cos \theta - \sin \phi) r^2 \sin \theta
\]
\[
= \frac{R^3}{3} \int_0^\frac{\pi}{2} d\theta \int_0^{2\pi} d\phi (5 \cos \theta \sin \theta - \sin \phi \sin \theta)
\]
\[
= \frac{R^3}{3} \int_0^{2\pi} d\phi \left[ \frac{5}{2} \sin^2 \theta + \cos \theta \sin \phi \right]_{\theta=0}^{\theta=\frac{\pi}{2}}
\]
\[
= \frac{R^3}{3} \int_0^{2\pi} d\phi (\frac{5}{2} - \sin \phi)
\]
\[
= \frac{5}{3} \pi R^3.
\]

For the right hand side we need two integrals. The surface element along the hemisphere is \( d\hat{a} = 4\pi r^2 \sin \theta d\theta d\phi \), with \( r = R \) everywhere along the surface. We also need to include the base, though. There we should hold \( \theta = \frac{\pi}{2} \), and our surface element is then \( d\hat{a} = r dr d\phi \). Thus,

\[
\oint_S \vec{v} \cdot d\hat{a} = \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi R^2 \sin \theta v_r + \int_0^R dr \int_0^{2\pi} d\phi rv_\theta
\]
\[
= \int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi R^3 \sin \theta \cos \phi + \int_0^R dr \int_0^{2\pi} d\phi r^2 \sin \theta
\]
\[
= \pi R^3 + \frac{2\pi R^3}{3} = \frac{5\pi R^3}{3}.
\]

**Problem 3** (Griffiths 1.42(c))

\[ \vec{v} = s(2 + \sin^2 \phi) \hat{s} + s \sin \phi \cos \phi \hat{\phi} + 3z \hat{z} \]

\[ \nabla \times \vec{v} = \hat{s} \left[ \frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] + \hat{\phi} \left[ \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] + \hat{z} \left[ \frac{1}{s} \frac{\partial sv_\phi}{\partial s} - \frac{\partial s}{\partial \phi} \right] \]
\[
= \hat{s} \left[ \frac{\partial}{\partial \phi} \left( 3z \right) - \frac{\partial}{\partial z} \left( s \sin \phi \cos \phi \right) \right] + \hat{\phi} \left[ \frac{\partial}{\partial z} \left( s(2 + \sin^2 \phi) \right) - \frac{\partial}{\partial s} \left( 3z \right) \right] + \hat{z} \left[ \frac{1}{s} \frac{\partial}{\partial s} \left( s^2 \sin \phi \cos \phi \right) - \frac{\partial}{\partial \phi} \left( s(2 + \sin^2 \phi) \right) \right]
\]
\[
= \hat{s} \left[ 0 - 0 \right] + \hat{\phi} \left[ 0 - 0 \right] + \hat{z} \left[ \frac{1}{s} \left( 2s \sin \phi \cos \phi - 2s \sin \phi \cos \phi \right) \right]
\]
\[
= 0
\]

2
Problem 4  (Griffiths 2.4)

From example 2.1 in the text we see that the electric field a distance \( r \) from a line of uniformly distributed charge of length \( 2L \) is

\[
\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{r \sqrt{r^2 + L^2}} \hat{r}
\]

where \( \hat{r} \) points directly away from the center of the line and perpendicular to it. So, to find the field a distance \( z \) from the center of the square loop shown in the figure we need to just sum up four such electric fields.

![Figure 2.8](image)

The distance from the point \( P \) to any of the four sides of the square will be \( r = \sqrt{z^2 + \left(\frac{a}{2}\right)^2} \). Each contribution to the electric field will have a component in the \( z \) direction as well as a component parallel to the plane of the square loop. However, these parallel components sum to zero because of the symmetry of the loop. Therefore, we just need to add up the \( z \) components to find the total \( \vec{E} \):

\[
\vec{E}(z) = 4 \left[ \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{r \sqrt{r^2 + L^2}} \right] \sin \theta_r \hat{z}
\]

\[
= \frac{1}{\pi\epsilon_0} \frac{2\lambda z}{r \sqrt{r^2 + \left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}} \hat{z}
\]

\[
= \frac{1}{\pi\epsilon_0} \frac{\lambda az}{\left(z^2 + \left(\frac{a}{2}\right)^2\right)^{3/2}} \hat{z}.
\]

Problem 5  (Griffiths 2.12)

Gauss’ Law for electric fields tells us

\[
\oint_S \vec{E} \cdot d\vec{A} = \int_V \vec{\nabla} \cdot \vec{E} dV = \frac{1}{\epsilon_0} \int_V \rho dV = \frac{q_{\text{encl}}}{\epsilon_0}
\]

regardless of the bounding surface. We should pick a surface (our ”Gaussian surface”) that uses the symmetry of the physical problem to our advantage. For a sphere of uniformly distributed charge the electric field on the surface of any concentric sphere will be radially symmetric. So, the surface integral of
the electric field over any such sphere will be the constant value of the field times the area of the sphere. The amount of charge enclosed by such a sphere is $q_{\text{encl}} = \rho \frac{4}{3} \pi r^3$, where $r$ is the radius of our Gaussian sphere ($r < R$).

\[ \oint_S \vec{E} \cdot d\vec{A} = \frac{q_{\text{encl}}}{\epsilon_0} \]

\[ E(r)4\pi r^2 = \frac{\rho \frac{4}{3} \pi r^3}{\epsilon_0} \]

So,

\[ \vec{E}(r) = \frac{\rho r}{3\epsilon_0} \hat{r}. \]

**Problem 6** (Griffiths 2.16)

For each region we should choose a cylinder as our Gaussian surface since the charge distribution is cylindrically symmetric. There will be no component of electric field in the axial direction, so our surface integral will only include the curved surface of whatever cylinder we pick (the ends are perpendicular to the field lines, so the surface integral is zero for them). For all three regions we will get

\[ \oint_S \vec{E} \cdot d\vec{A} = \frac{q_{\text{encl}}}{\epsilon_0} \]

\[ E(s)2\pi sl = \frac{q_{\text{encl}}}{\epsilon_0} \]

where $l$ is the length (arbitrary) of whatever Gaussian cylinder we construct.

\[ s < a \]
Inside the inner cylinder the charge enclosed depends on the radius $s$ of our Gaussian cylinder as $q_{\text{encl}} = \rho s^2 l$. Gauss' Law becomes

$$E(s)2\pi sl = \frac{\rho s^2 l}{\epsilon_0}$$

so

$$\vec{E}(s) = \frac{\rho s}{2\epsilon_0} \hat{s}.$$ 

ii $a < s < b$

Between the inner cylinder and outer shell the charge enclosed by the coaxial Gaussian cylinder is the total charge of a length $l$ of the inner cylinder.

$$E(s)2\pi sl = \frac{\rho \alpha^2 l}{\epsilon_0}$$

so

$$\vec{E}(s) = \frac{\rho \alpha^2}{2\epsilon_0 s} \hat{s}.$$ 

iii $s > b$ Everywhere outside the outer shell the total charge enclosed by a coaxial Gaussian cylinder is always zero. Therefore, $\vec{E}(s) = 0$.

Here is the plot (in arbitrary units) of $|\vec{E}(s)|$ with $a = 1$ and $b = 3$: 

![Graph of electric field modulus](image-url)
**Problem 7** (Griffiths 2.17)

Since charge is distributed infinitely in the $xz$-plane we know the electric field will only be a function of $y$. We can choose a variety of Gaussian surfaces, but whatever surface we choose should have boundaries that are parallel and perpendicular to the $xz$-plane. Let’s use a "pillbox" (as it is called in the text) centered at the origin with area $A$ parallel to the $xz$-plane. In the $y$ direction our surface should have dimension $2y$. This insures that the electric field strength is uniform over two surfaces (the two perpendicular to the $y$-axis), and the surface integral over the other four surfaces is zero. Therefore, Gauss’ Law will give us

$$E(y)2A = \frac{q_{encl}}{\epsilon_0}.$$  

**i** $y < d$

For a Gaussian pillbox inside the slab the charge enclosed is $q_{encl} = \rho A 2y$. We have

$$E(y)2A = \frac{\rho A 2y}{\epsilon_0}$$

so

$$\vec{E}(y) = \frac{\rho y}{\epsilon_0} \hat{y}$$

**ii** $y > d$

Outside the slab our Gaussian pillbox encloses an amount of charge limited by the thickness of the slab: $q_{encl} = \rho A 2d$. Therefore,

$$E(y)2A = \frac{\rho A 2d}{\epsilon_0}$$

so

$$\vec{E}(y) = \frac{\rho d}{\epsilon_0} \hat{y}$$

Here is a plot of the electric field due to the slab. The vertical axis is the electric field strength in units of $\frac{\rho d}{\epsilon_0}$ and the horizontal axis is the distance in
the $y$ direction in units of $d$.

Note the similarities among the results of Problems 5, 6 and 7. In each case the electric field inside a uniform charge distribution is linear in some coordinate. Outside the charge distribution the field weakens differently depending on the geometry of the situation.