Problem 1  (Griffiths 5.15)
From Example 5.9 we have the field from a single solenoid:

\[ \vec{B} = \mu_0 nI \hat{z} \text{ inside the solenoid (it's zero outside)} \]

The magnetic field is a vector field, so it must obey the principle of superposition. So the field in each region just depends on which solenoid encloses the region.

i  \hspace{1em} s < a

\[ \vec{B} = \mu_0 (n_1 - n_2) I \hat{z} \]

where \( \hat{z} \) is the direction of the field due to the inner conductor (to the left in Figure 5.42).

ii  \hspace{1em} a < s < b

\[ \vec{B} = -\mu_0 n_2 I \hat{z} \]

iii  \hspace{1em} s > b

\[ \vec{B} = 0 \]

Problem 2  (Griffiths 5.16 - optional)
The plates are large and the charge densities are uniform. Let the plates be parallel to the \( x-y \) plane and moving in the \( \hat{x} \) direction. The surface current is then uniform at \( z = \pm \frac{d}{2} \). We have

\[ \vec{K}(z) = \left( \delta(z - \frac{d}{2}) - \delta(z + \frac{d}{2}) \right) \sigma v \hat{x}. \]

We know how to find the field from a surface current. From Example 5.8 we see that the field from a uniform surface current in the positive \( \hat{x} \) direction is uniform and in the \( -\hat{y} \) direction above the plane and in the \( \hat{y} \) direction below the plane. This is reversed, of course for a current in the negative \( \hat{x} \) direction. With this in mind we can easily answer the following questions.
(a) Above and below the plates the field is zero. This can be seen from the above description as well as by a straightforward application of Ampere’s law (the enclosed current is zero for any loop that intersects both sheets of charge). Between the plates the fields from both plates are in the positive $\hat{y}$ direction. Thus, the total field is

$$\vec{B} = 2 \left( \frac{\mu_0 K}{2} \hat{y} \right) = \mu_0 \sigma v \hat{y}.$$ 

(b) The force on the upper plate will be the force on a current density $K \hat{x}$ due to the magnetic field from the lower plate:

$$\vec{F}_B = \vec{K} \times \vec{B} = K \frac{\mu_0 K}{2} (\hat{x} \times \hat{y}) = \frac{\sigma^2 v^2 \mu_0}{2} \hat{z} \quad \text{(away from the other plate)}.$$

(c) The force per unit area due to the electrostatic forces is just the electrostatic pressure that we have learned. Thus, for the electrostatic forces to balance the magnetic forces we require

$$\frac{\vec{F}_E}{A} = \frac{\vec{F}_B}{A} \quad \text{and} \quad \frac{\sigma^2}{2 \varepsilon_0} = \frac{\sigma^2 v^2 \mu_0}{2} \quad \text{(the speed of light)}.$$

Note: It turns out, the speed of light is closely tied to electrodynamic phenomena, as you will see in the next chapters. What you may not learn until later in your physics career is that the electric and magnetic fields are really even more closely related than they seem. Electric fields experienced by charges moving at speeds close the the speed of light have a magnetic component, and vice versa.

**Problem 3** (Griffiths 5.24)

$$\vec{\nabla} \cdot \vec{A} = -\frac{1}{2} \vec{\nabla} \cdot (\vec{r} \times \vec{B}) = -\frac{1}{2} \left( \vec{B} \cdot (\vec{\nabla} \times \vec{r}) - \vec{r} \cdot (\vec{\nabla} \times \vec{B}) \right) = 0 \quad \text{(for constant $\vec{B}$)},$$

and

$$\vec{\nabla} \times \vec{A} = -\frac{1}{2} \vec{\nabla} \times (\vec{r} \times \vec{B}) = -\frac{1}{2} \left( (\vec{B} \cdot \vec{\nabla}) \vec{r} - (\vec{r} \cdot \vec{\nabla}) \vec{B} + \vec{r}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{r}) \right).$$
Thus, $\vec{A} = -\frac{1}{2} (\vec{r} \times \vec{B})$ clearly works as a vector potential for constant $\vec{B}$. Certainly there are many others. For example, we could add any constant vector to $\vec{A}$. Also, we could add any vector function whose curl and divergence both vanish. From the second condition, however, it is clear that we could not multiply $\vec{A}$ by an arbitrary constant.

**Problem 4** (Griffiths 5.26)

You may be tempted, as I was, to write $\vec{A} = \frac{\mu_0}{4\pi} \int \vec{K} \, da'$. However, $\vec{K}$ does not go to zero at infinity in this case, so we cannot use the patent solution to Laplace’s equation here. It is relatively straightforward, though, to find $\vec{A}$ from $\vec{B}$. We know

$$\vec{B}(\pm |z|) = \mp \frac{\mu_0 K}{2} \hat{y}$$

and so we can write

$$\vec{\nabla} \times \vec{A} = \frac{\partial A_y}{\partial z} \hat{y} - \frac{\partial A_z}{\partial y} \hat{z} = \vec{B}$$

(since $\vec{A}$ must point in the $\hat{x}$ direction because that’s the only direction of $\vec{K}$)

$$\Rightarrow \vec{A} = \pm \frac{\mu_0 K}{2} z \hat{x}$$

(because $\vec{A}$ cannot depend on $y$).

Since this $\vec{A}$ satisfies all the appropriate conditions, we can use it (even though it is not unique).

**Problem 5** (Griffiths 5.34)

a

$$\vec{m} = \pi R^2 I \hat{z}$$

b

$$\vec{B}(r, \theta) = \frac{\mu_0 R^2 I}{4r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta})$$
c  On the $z$ axis, the result from (b) gives
\[ \vec{B}(z) = \frac{\mu_0 R^2 I}{2z^3} \hat{\imath} \]
and from Example 5.6 we have
\[ \lim_{z \to \infty} \vec{B}(z) = \frac{\mu_0 I R^2}{2} \left( -\frac{3}{z^3} \left( 1 + \frac{R^2}{z^2} \right)^{-2} \right) \hat{\jmath} \]
\[ \vec{B}(z) \approx \frac{\mu_0 R^2 I}{2z^3} \hat{\jmath}. \]

**Problem 6**  (Griffiths 5.35 - optional)

Breaking the record into rings of radius $r$ and current density $\sigma \omega r$ we have
\[ \vec{m} = \int_0^R \sigma \omega rdr \pi r^2 \hat{\omega} \]
\[ = \frac{\sigma \omega \pi R^4}{4} \hat{\omega}. \]

**Problem 7**  (Griffiths 6.1 & 6.3)

6.1 Let $\vec{r}'$ be $\hat{x}$ for simplicity. Then $\vec{m}_1$ points in the $\hat{z}$ direction. Thus,
\[ \vec{r} = \vec{m}_2 \times \vec{B}_{m_1} = Ib^2 \hat{r} \times \frac{\mu_0 I \pi a^2}{4 \pi x^3} (\hat{z}) \]
\[ = \frac{\mu_0 I^2 a^2 b^2}{4 \pi ^3} \hat{y} \] (toward the $-z$ axis).

Thus, the second dipole anti-aligns with the first. This is why ferro-magnetism is so interesting - it doesn’t follow this rule!

6.3 Using Griffiths’ Equation 6.2 we should write $\vec{B}$ due to $m_1$ and compute the force due to that field on the second dipole, $m_2$, consisting of an ideal loop of current $I$ and radius $R$. The field at the edge of the loop that makes up $m_2$ will be
\[ \vec{B}(r, \theta) = \frac{\mu_0 m_1}{4\pi r^3} \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right) \]
where $\vec{r}$ is the vector pointing from $m_1$ (say, at the origin) to a point on the loop that comprises $m_2$ (see Figure 1). Now, let $\vec{r}$ lie in the $y - z$ plane for
simplicity. Then we may write

\[ \vec{B}(r, \theta) = \frac{\mu_0 m_1}{4\pi r^3} \left[ 2 \cos \theta \left( \sin \theta \hat{\imath} + \cos \theta \hat{k} \right) + \sin \theta \left( \cos \theta \hat{\imath} - \sin \theta \hat{k} \right) \right] \]

\[ = \frac{\mu_0 m_1}{4\pi r^3} \left( 3 \cos \theta \sin \theta \hat{\imath} + (2 \cos^2 \theta - \sin^2 \theta) \hat{k} \right) \]

Now, our force according to Equation 6.2 is

\[ F = 2\pi iR B \cos \theta' \]

where \( \theta' \) is the angle between the \( \vec{B} \) field and the plane of the loop. Thus, as we have arranged it we know that

\[ B \cos \theta' = \vec{B} \cdot \hat{\imath} \]

\[ \implies F = 2\pi iR \vec{B} \cdot \hat{\imath} \]

\[ = 2\pi iR \frac{\mu_0 m_1}{4\pi r^3} \left( 3 \cos \theta \sin \theta \right). \]

Replacing the \( \cos \theta \) and \( \sin \theta \) with the appropriate ratios, and recognizing that \( r \gg R \), gives

\[ F = 2\pi iR \frac{\mu_0 m_1}{4\pi r^3} \left( \frac{3 \sqrt{r^2 - R^2} R}{r} \right) \]

\[ = i\pi R^2 \frac{3 \mu_0 m_1}{2\pi r^4} \]

\[ = \frac{3\mu_0 m_1 m_2}{2\pi r^4}. \]

Of course, the force is attractive, so

\[ \vec{F} = -\frac{3\mu_0 m_1 m_2}{2\pi r^4} \hat{z}. \]
Using Equation 6.3 is much easier. Since $\vec{m}_1$ and $\vec{m}_2$ both point in the same direction we have

$$\vec{F} = \vec{\nabla} (\vec{m}_2 \cdot \vec{B}_1)$$

$$= \vec{\nabla} \left( m_2 \hat{r} \cdot \frac{\mu_0 m_1}{4\pi r^3} (2\hat{r}) \right)$$

$$= \vec{\nabla} \left( \frac{\mu_0 m_1 m_2}{2\pi r^3} \right)$$

$$= -\frac{3\mu_0 m_1 m_2}{2\pi r^4} \hat{r}$$

**Problem 8** (Griffiths 6.7)

The bound currents are

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$$

and

$$\vec{K}_b = \vec{M} \times \hat{n} = M \hat{\phi}.$$ 

Thus, the uniformly magnetized cylinder acts like a solenoid with current running in the circumferential direction. The field of such a solenoid is

$$\vec{B}(r) = \begin{cases} 
\mu_0 M \hat{z} & \text{inside} \\
0 & \text{outside.}
\end{cases}$$