**Final Exam, Electrodynamics 1: 7110**

**Problem 1**.

A spaceship identifies distances to various objects and velocities of those objects by emitting electromagnetic sensing pulses of fixed frequency and then detecting (1) frequencies of the returned signals reflected by the objects, and (2) time delays before the reflected signals are returned to the spaceship.

a) An object is traveling with velocity $V$ along the $x$-axis of the reference frame of the spaceship. The spaceship emits a sensing pulse of frequency $\omega_0$; the signal propagating along the positive direction of the $x$-axis. What is the frequency of the sensing pulse $\omega'$ in the reference frame of the object?

b) An observer on the spaceship detects that the signal reflected from the object has frequency $\omega_R$. Determine velocity $V$ from $\omega_0$ and $\omega_R$. (Do not assume that $V$ is small compared with the speed of light.)

c) The observer detects that time $t$ passes after the sensing pulse is emitted and before the reflected signal is returned. Determine the distance from the spaceship to the object at the instance of time when the reflected signal arrives at the spaceship.

**Problem 2**.

A *non-conducting* spherical shell of radius $a$ is charged with a fixed surface charge density $\sigma(\theta) = \sigma_0 \cos\theta$ ($\theta$ is the polar angle of the spherical system of coordinates). The non-conducting shell is surrounded by a concentric spherical *conducting* shell of radius $2a$.

a) Identify the functional form of the electric potential $\varphi$ in the region between the two shells, $a \leq r \leq 2a$.

b) Determine the electric potential $\varphi$ in this region by satisfying appropriate boundary conditions.

c) A negatively charged electron (charge $-e$) of mass $m$ leaves (with zero initial velocity) the surface of the conducting shell at a location where the density of the induced negative charges is the largest. Determine the time it takes the electron to reach the non-conducting shell. Note that

$$\int\limits_{1/2}^{1} \frac{x\,dx}{\sqrt{1-x^3}} \approx 0.73.$$

**Problem 3**.

A point quadrupole produces the electric potential ($\theta$ is the polar angle of the spherical system of coordinates)

$$\varphi = \frac{Q_{zz}}{16\pi\varepsilon_0}\frac{3\cos^2\theta - 1}{r^3}. \tag{1}$$

a) Find all non-zero components of the electric field of the quadrupole in the spherical coordinates.

b) The quadrupole moment $Q_{zz}$ changes linearly with time $dQ_{zz}/dt = \alpha$. Assuming that the induced magnetic field has only the azimuthal component $B_\phi$, write all components of the equation for the curl of the induced magnetic field,

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

c) Solve the components of the above equation and find $B_\phi(r,\theta)$. Demonstrate that the obtained solution satisfies all components of the above equation.

d) Show whether the solution you found satisfies the equation $\nabla \cdot \mathbf{B} = 0$.

For reference, in a spherical system of coordinates $r$, $\theta$, $\phi$, the components of $\nabla \times \mathbf{A}$ and $\nabla \cdot \mathbf{A}$ are

$$(\nabla \times \mathbf{A})_r = \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi},$$

$$(\nabla \times \mathbf{A})_\theta = \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r},$$

$$(\nabla \times \mathbf{A})_\phi = \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta},$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}.$$