Graduate Comprehensive Exam 2020 - Quantum Mechanics

December 7, 2020

1. (36 pts) A nonrelativistic particle of mass m moves in the presence of a 3D isotropic harmonic oscillator potential of frequency ω. We define creation operators $a_1^\dagger$, $a_2^\dagger$, and $a_3^\dagger$ and annihilation operators a_1, a_2, and a_3 for the three Cartesian directions.

   (a) (6 pts) Write the hamiltonian in terms of these operators.

   (b) (6 pts) What is the energy and degree of degeneracy of the ground state?

   (c) (6 pts) What is the energy and degree of degeneracy of the first excited state?

   (d) List the kets for the first excited state in terms of these operators and the ground state ket $|0,0,0\rangle$.

   (e) (4 pts) We consider the operator

   $$L_i = \sum_{j,k} i\hbar\epsilon_{ijk} a_j a_k^\dagger \,. \tag{1}$$

   Show that it commutes with the hamiltonian.

   (f) (4 pts) Write the commutation relation that the components of $\mathbf{L}$ must satisfy if it is to be an angular momentum operator.

   (g) (4 pts) Show that, indeed, $\mathbf{L}$ satisfies the commutation relations. (It suffices to choose two different components of $\mathbf{L}$ and show it for them.)

   (h) (4 pts) The operator $\mathbf{L}^2$, based on the above definition, is

   $$\mathbf{L}^2 = \hbar^2 \left[N(N+1) - \sum_{j,k} a_k^\dagger a_k^\dagger a_j a_j \right] \,,$$

   where $N = n_1 + n_2 + n_3$ counts the number of oscillator quanta. Use this expression to determine the angular momentum quantum number ℓ for the first excited state.

   (i) (4 pts) Find the linear combinations of first-excited-state kets that form eigenstates of L_3 for each possible eigenvalue. Normalize them.

2. (36 pts) A canonical ensemble of electrons is in a uniform external magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The hamiltonian is

$$H = -\frac{e\hbar}{m_e c}\mathbf{S} \cdot \mathbf{B}\,.$$

The ensemble is in thermal equilibrium at temperature T.

(a) (6 pts) What are the energy levels of an electron in the ensemble?

(b) (6 pts) Write the density matrix that describes the ensemble in the eigenbasis of the z component of the spin, S_z.

(c) (6 pts) If an electron is chosen at random from the ensemble and S_z is measured, what is the probability that the result is $+1/2\hbar$?

(d) (4 pts) What is the expectation value of the vector spin operator $\mathbf{S}$?

(e) (4 pts) The external magnetic field is instantaneously changed so that it points in the positive y direction with the same strength. This is done so that immediately afterwards, the density matrix has not yet changed. The following questions apply to the new field. Write the matrix elements of the time-evolution operator for the new hamiltonian in the eigenbasis of S_z. Let $t = 0$ be the time the field is changed.

(f) (4 pts) We are interested in the short-time behavior of the system, long before thermal equilibrium is restored. That is, ignore the thermal bath. Describe briefly in semiclassical terms the motion of the spins of the particles in the ensemble.

(g) (4 pts) If a particle is chosen at random from the ensemble and S_z is measured, what is the probability as a function of time that the result is $+1/2\hbar$?

(h) (4 pts) How does the expectation value of the vector spin operator $\mathbf{S}$ vary during the same short time period?

3. (28 pts) The probability of an electric dipole transition in a hydrogenic atom is given by the square of the transition amplitude $\langle n', j', m' | e\mathbf{r} | n, j, m\rangle$ where $\mathbf{r} = (x, y, z)$ is the coordinate of the electron, and n, j, and m give the principal quantum number, the total angular momentum, and the total magnetic quantum number of the electron orbital.

(a) (6 pts) Express x, y, and z in terms of spherical tensors. Be sure to specify the rank and component and define your notation.

(b) (6 pts) For a given initial state j, what are the allowed values of j'?

(c) (6 pts) For the z component of the electric dipole operator, and for $m = j$, what are the possible values of m'?

(d) (6 pts) For the x component and for $m = j$, what are the possible values of m'?

(e) (4 pts) Find the ratio of the transition amplitudes for the z and x components when $m = j$. Express it in terms of the appropriate Clebsch-Gordon coefficients.