Problem 1 [20 pts]

A thin square loop $C$ of sides $2b$ as shown lies in the $xy$-plane. It is non-conducting and embedded with a uniform linear charge density $\lambda$. The loop is centered on, and is spinning about the $z$-axis at a constant angular speed $\omega$ in the counter-clock-wise sense as shown (i.e. $\vec{\omega} = \omega \hat{z}$). At the time shown, its sides are parallel to the $x$-axis and $y$-axis.

Find the magnetic (dipole) moment $\vec{m}$ of this spinning loop following the steps below.

Clearly the loop $C$, can be considered to be the sum of 4 simple paths $C_1, C_2, C_3, C_4$, that comprise the 4 sides of the square as shown. So that the dipole moment $\vec{m} = \vec{m}_1 + \vec{m}_2 + \vec{m}_3 + \vec{m}_4$ is the just sum of the four parts. There is clearly symmetry over the four segments.

We will start by considering $\vec{m}_1$ from the front segment $C_1$. Make all your calculations at the time shown.

*** Note that this problem does not reduce to a linear current circulating in the square loop.

(a) [3 pts] For the line element $ds'_1$ (note these are scalars in this problem) on $C_1$ located at $y'$ to the right of the $x$-axis as shown, write down the position vector $\vec{r'}$ in terms of Cartesian coordinates ($x', y'$, and/or $z'$) plus given constants, AND in Cartesian components (i.e. as a linear combination of unit vectors $\hat{x}, \hat{y}$, and/or $\hat{z}$).

*** Note we are using primed coordinates as usual to indicate position within a charge/current distribution that generates an electric/magnetic field.

(b) [3 pts] Calculate the instantaneous velocity $\vec{v'}$ of the line element $ds'_1$, using the given $\vec{\omega}$ and the position vector $\vec{r'}$ in Cartesian coordinates and Cartesian components.

(c) [3 pts] Write the (3-scalar) line element $ds'_1$ in Cartesian coordinates (and/or their differentials), and hence write down the charge element $dq'$ in terms of Cartesian coordinates (and/or their differentials), and given constants.

(d) [4 pts] Calculate the integrand for $\vec{m}_1$ that involves a further cross-product. Use Cartesian coordinates and components. Remember cross-products are vectors.

(e) [4 pts] Integrate over the straight line segment $C_1$ at the time shown (parallel to the $y$-axes) to find the magnetic moment $\vec{m}_1$. Hint: remember $\vec{m}_1$ is a vector.

(f) [3 pts] Use the symmetry of the system and you answer for $\vec{m}_1$ to write down solutions for $\vec{m}_2, \vec{m}_3$, and $\vec{m}_4$. Hence find the total dipole moment $\vec{m}$. Remember they are all VECTORS!!!
We note that \( \overrightarrow{m} = \overrightarrow{m}_1 + \overrightarrow{m}_2 + \overrightarrow{m}_3 + \overrightarrow{m}_4 \)

\[ C \quad C_1 \quad C_2 \quad C_3 \quad C_4 \]

We will start by looking at \( C_1 \): run \( x \) to \( y \)-axis

(a) \( d\overrightarrow{s}_1 = dy' \)

(y': from \(-b\) to \(+b\)

\[ \lambda \overrightarrow{\theta}_i = \lambda d\overrightarrow{s}_1 = \lambda dy' \]

(b) location \( \overrightarrow{r}' = bx' + y'y' \) \((z' = 0)\)

(c) So for that line element

\( \overrightarrow{\nabla}' = \overrightarrow{\omega} \times \overrightarrow{r}' = \overrightarrow{\omega} \times (bx' + y'y') \)

\[ \overrightarrow{\nabla}' = \overrightarrow{\omega} (-y'y' \hat{x} + bx') \]

(d) \( \overrightarrow{m}_1 = \frac{1}{2} \int_{C_1} d\overrightarrow{\theta}_i (\overrightarrow{r}' \times \overrightarrow{\nabla}') = \frac{1}{2} \int_{C_1} d\overrightarrow{\theta}_i (\overrightarrow{r}' \times \overrightarrow{\nabla}') \)

\( \overrightarrow{r}' \times \overrightarrow{\nabla}' = (bx' + y'y') \times \overrightarrow{\omega} (-y'y' \hat{x} + bx') \)

\[ \overrightarrow{r}' \times \overrightarrow{\nabla}' = \overrightarrow{\omega} (b^2 + y'^2) \hat{z} \]

(e) \( \overrightarrow{m}_1 = \frac{1}{2} \int_{C_1} d\overrightarrow{\theta}_i (\overrightarrow{r}' \times \overrightarrow{\nabla}') = \frac{1}{2} \int_{-b}^{b} dy' \cdot \overrightarrow{\omega} (b^2 + y'^2) \hat{z} \)

--- Cont'd
(e) cont'd

\[ \bar{m}_1 = \frac{\lambda \omega^2 x}{2} \int_{-b}^{b} \left( b^2 + y' \right) dy' = \frac{\lambda \omega^2 x}{2} \left[ b \left( \frac{b^2}{3} + \frac{y'^2}{3} \right) \right]_{-b}^{b} \]

\[ = \frac{x}{2} \frac{\lambda \omega^2}{2} \left[ b^3 + \frac{b^3}{3} - \left( -b \right)^3 - \left( -b \right)^3 \right] \]

\[ = \frac{x}{2} \frac{\lambda \omega^2}{2} \cdot \frac{8b^3}{3} = \frac{4 \lambda \omega b^3 \Delta}{3} \]

(f) Note \( \bar{m}_1 \) points only in the \( \hat{z} \) direction.

\[ \Rightarrow \] by symmetry

\[ \bar{m}_2 = \bar{m}_3 = \bar{m}_4 = \bar{m}_1 = \frac{4 \lambda \omega b^3 \Delta}{3} \]

\[ \Rightarrow \bar{m} = \frac{16 \lambda \omega b^3 \Delta}{3} \]
Problem 2 [20 pts]

A hollow, non-conducting, thin sphere of radius $a$ is centered on the origin. The surface (at radius $a$) is held at a non-isotropic potential given by

$$\varphi(a, \theta, \phi) = V_0 \sin \theta \cos \phi$$

Follow the steps below to find the potential in the empty interior (i.e. $r < a$) of the sphere. Note that $\theta$ is the polar angle measured from the $+z$-axis, while $\phi$ is the azimuthal angle measured in the $xy$-plane counterclockwise from the $+x$-axis.

(a) [5 pts.] Write down the most general solution $\varphi(r, \theta, \phi)$, to the Laplace Equation $\nabla^2 \varphi = 0$, when solved by separation of variables in spherical coordinates $r, \theta, \phi$. This should be an infinite series summing over two indices, $l$ and $m$. As we have done in class, use the coefficients $A_{lm}$ for the non-negative (zero or positive) powers of $r$ and $B_{lm}$ for the negative powers of $r$.

(b) [5 pts.] First apply the implicit boundary condition that the value of $\varphi$ is finite at the origin. This should eliminate half of the coefficients (i.e. they are all zero for all values of $l$ and $m$). Indicate which coefficients vanish from this boundary condition and write the new, now restricted general solution for $r < a$.

(c) [10 pts] Now apply the stated boundary condition at $r = a$. Solve for the coefficients for $l = 0$ and $l = 1$, and all allowed values of $m$ thereof.

Spherical Harmonics:

$$Y_0^0(\theta, \phi) = \frac{1}{2} \frac{1}{\sqrt{\pi}}$$

$$Y_1^{-1}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{-i\phi}$$

$$Y_1^{0}(\theta, \phi) = \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos \theta$$

$$Y_1^{1}(\theta, \phi) = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{i\phi}$$
\[ \Phi(a, \theta, \phi) = \frac{V_0 \sin \phi \cos \phi}{a} \]

Find \( \Phi(r, \theta, \phi) \) in the region 
\( r < a \)

Most General Solution:

(a) 
\[ \Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ A_{lm} r^{-l} + B_{lm} r^{-l+2} \right] Y_l^m(\theta, \phi) \]

(b) boundary condition @ \( r = 0 \)
\( r \to 0 \), \( r \to \infty \)

But \( \Phi(r, \theta, \phi) \) should be finite:
\[ \Rightarrow [B_{lm} = 0] \text{ for all } l, m. \]

\[ \Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} r^{-l} Y_l^m(\theta, \phi) \]

(c) 
\[ \Phi(a, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} a^{-l} Y_l^m(\theta, \phi) = \frac{V_0 \sin \theta \cos \phi}{a} \]

Now we take \( l = m \):
\[ \int_0^{2\pi} \int_0^\pi Y_l^m(\theta, \phi) \Phi(a, \theta, \phi) \sin \theta d\theta d\phi \]
\[ = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} a^{-l} \int_0^{2\pi} \int_0^\pi Y_l^m(\theta, \phi)^2 \sin \theta d\theta d\phi \]
\[ = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} a^{-l} \delta_{lm} \delta_{m2} = a^2 A_{lm} \]

\[ A_{lm} = \frac{1}{a^2} \int_0^{2\pi} d\phi \int_0^\pi \sin^2 \theta Y_l^m(\theta, \phi)^2 \sin \theta d\theta \]

\[ \frac{V_0}{a^2} \int_0^{2\pi} d\phi \int_0^\pi \sin^2 \theta \sin \theta \cos \phi \]
\( l = 0 \), \( m = 0 \):
\[
Y^m_l(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi (2l)!}} P_l^m(\cos\theta) e^{\pm il\phi}
\]
\[
A_{00} = \frac{V_0}{a^2} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sqrt{\frac{1}{4\pi}} \sin^2 \theta \cos \phi
\]
\[
= \frac{V_0}{a} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin^2 \theta \cos \phi
\]
\[
A_{00} = 0
\]

\( l = 1 \), \( m = 0 \):
\[
Y^m_l(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi (l+1)!}} P_l^m(\cos\theta) e^{\pm il\phi}
\]
\[
A_{10} = \frac{V_0}{a^2} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sqrt{\frac{2}{4\pi}} \cos \theta \sin^2 \theta \cos \phi
\]
\[
= \frac{V_0}{a} \int_0^{2\pi} d\phi \int_0^\pi d\theta \cos \theta \sin^2 \theta \cos \phi
\]
\[
A_{10} = 0
\]

\( l = 1 \), \( m = \pm 1 \):
\[
Y^m_l(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi (l+1)!}} (1 \pm \frac{1}{2}) P_l^m(\cos\theta) e^{\pm il\phi}
\]
\[
A_{1\pm 1} = \frac{V_0}{a^2} \int_0^{2\pi} d\phi \int_0^\pi d\theta \left[ -\frac{\sqrt{2}}{8\pi} \sin \theta \cos \phi \right] e^{\pm il\phi}
\]
\[
= -\frac{V_0}{a} \int_0^{2\pi} d\phi \int_0^\pi d\theta \left[ -\frac{\sqrt{2}}{8\pi} \sin \theta \cos \phi \right]
\]
\[
A_{1\pm 1} = \frac{V_0}{a} \int_0^{2\pi} d\phi \int_0^\pi d\theta \left[ -\frac{\sqrt{2}}{8\pi} \sin \theta \cos \phi \right]
\]
\[
= -\frac{V_0}{a} \int_0^{2\pi} d\phi \int_0^\pi d\theta \left[ -\frac{\sqrt{2}}{8\pi} \sin \theta \cos \phi \right]
\]
\[
\int_0^{\pi} \cos^2 \phi \, d\phi = \pi, \quad \int_0^{\pi} \sin \phi \cos \phi \, d\phi = \frac{1}{2} \sin \phi \bigg|_0^{\pi} = 0
\]

\[
\int_0^{\pi} \sin^2 \phi \, d\phi = - \int_0^{\pi} \left( \cos^2 \phi \right) \, d\phi = - \left[ \cos \phi - \frac{1}{2} \sin^2 \phi \right]_0^{\pi} = - \left( -1 + \frac{1}{2} \right) = \frac{1}{2}
\]

Therefore,

\[
A_{11} = \frac{\sqrt{3}}{a} \sqrt{\frac{2}{8 \pi}} \cdot \pi \cdot \frac{4}{3} = -\sqrt{\frac{3}{2}} \frac{V_0}{a}
\]

\[
Y_{k-1}^l (\rho, \phi) = \left[ \sqrt{\frac{3}{8 \pi}} \sin \theta \, e^{i \phi} \right] \Rightarrow \sqrt{\frac{3}{8 \pi}} \sin \theta \, e^{i \phi}
\]

\[
A_{11} = \frac{V_0}{a^2} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left[ \sqrt{\frac{3}{8 \pi}} \sin^2 \theta \cdot \cos \phi (\cos \phi + i \sin \phi) \right]
\]

\[
= \frac{V_0}{a} \int_0^{2\pi} \cos \phi \left( \cos^2 \phi + i \sin \phi \cos \phi \right) d\phi \int_0^{\pi} \sin^2 \theta \, d\theta
\]

\[
= \frac{V_0}{a} \sqrt{\frac{3}{8 \pi}} \cdot \pi \cdot \frac{4}{3}
\]

\[
A_{11} = \frac{V_0}{a} \sqrt{\frac{2}{8 \pi}} \cdot \pi \cdot \frac{4}{3} = \sqrt{\frac{2a}{3}} \frac{V_0}{\sqrt{3}}
\]
Problem 3 (20 pts)

The figure to the right shows the standard method of obtaining a tune-able gamma ray beam using the process of Inverse Compton Scattering. A beam of electron of original (magnitude of 3-) momentum \( p_0 \) and rest mass \( m \) travels in the positive +x direction along the x-axis. A laser beam of original wavelength \( \lambda_0 \) travels in the +y direction along the y-axis. The two beams collide elastically (i.e. the electron stays an electron) at the origin. The recoiling photon gets energy and momentum from the electron and now has new energy \( E \), and travels at angle \( \theta \). The electron is scattered with (magnitude of 3-) momentum \( p \), at angle \( \phi \), where both angles \( \theta \) and \( \phi \) are measured counter-clock-wise from the +x axis in the xy-plane).

By selecting the photons (and placing a collimator) at a particular angle \( \theta \), you can choose the energy \( E \) of the new photon beam. This problem asks you to find \( E \) in terms of \( \theta \), the parameters \( \lambda_0, p_0 \), and the natural constants \( m, h \) and \( c \), following the steps below.

(a) [4 pts] Write down the 4-momenta, \( P^\mu_e \) and \( P^\mu_\gamma \) of the electron and photon BEFORE the collision in terms of \( \lambda_0, p_0, m, h \) and \( c \): i.e. in the form (replace the elements shown):

\[
P^\mu_e = \begin{bmatrix} E_e / c \\ p^e_x \\ p^e_y \\ p^e_z \end{bmatrix}, \quad P^\mu_\gamma = \begin{bmatrix} E_\gamma / c \\ p^\gamma_x \\ p^\gamma_y \\ p^\gamma_z \end{bmatrix}
\]

(b) [4 pts] Write down the 4-momenta, \( P'^\mu_e \) and \( P'^\mu_\gamma \) of the electron and photon AFTER the collision in terms of \( E, p, m, h, c, \theta \) and \( \phi \): i.e. in the form (replace the elements shown):

\[
P'^\mu_e = \begin{bmatrix} E'_e / c \\ p'_x \\ p'_y \\ p'_z \end{bmatrix}, \quad P'^\mu_\gamma = \begin{bmatrix} E'_\gamma / c \\ p'_\gamma_x \\ p'_\gamma_y \\ p'_\gamma_z \end{bmatrix}
\]

*** Note in this problem, the “prime” (i.e. apostrophe) indicates quantities AFTER the collision, NOT those in a moving frame \( S' \).

(c) [4 pts] Write down three independent equations involving \( E, p, \theta \) and \( \phi \) on the left hand side (LHS), and involving \( \lambda_0, p_0 \) on the right hand side (RHS), of each equation.

(d) [4 pts] Use two of the equations to eliminate \( \phi \) and solve for \( p^2 \) (magnitude squared of the three – vector momentum) in terms of \( \theta, E, p_0, m, h \) and \( c \), then substitute your expression for \( p^2 \) into (and eliminate \( p \), completely, from) the remaining equation.

(e) [4 pts] Algebraically solve for \( E \) in the remaining equation in terms of \( \lambda_0, p_0, m, h, c \), and \( \theta \).
1(a) Incident electrons of momentum $p_e = p_0$

$$E_0/c = \sqrt{p_e^2 + m^2c^2} = \sqrt{p_0^2 + m^2c^2}$$

$P_e =$

$$\begin{bmatrix}
\sqrt{p_e^2 + m^2c^2}  \\
p_0 \\
0 \\
0 \\
0
\end{bmatrix}$$

Original photon: $\lambda_0 : E_\gamma = \frac{hc}{\lambda_0}, \quad p_\gamma = \frac{h}{\lambda_0}$

$P_\gamma =$

$$\begin{bmatrix}
h/\lambda_0 \\
0 \\
h/\lambda_0 \\
0 \\
0
\end{bmatrix}$$

(b) $e^-$ with new (3-vector magnitude) momentum $p_e' = p$

$$E_e'/c = \sqrt{p_e'^2 + m^2c^2} = \sqrt{p^2 + m^2c^2}$$

$P_e' =$

$$\begin{bmatrix}
\sqrt{p^2 + m^2c^2} \\
p \cos \phi \\
p \sin \phi \\
0 \\
0
\end{bmatrix}$$

New photon with $E_\gamma' = E, \quad \Rightarrow p_\gamma' = E/c$

$P_\gamma' =$

$$\begin{bmatrix}
E/c \\
E/c \cos \theta \\
E/c \sin \theta \\
0 \\
0
\end{bmatrix}$$
1(c) 4-vector (total momentum) is conserved

$$\Rightarrow P_{E}^{\mu} + P_{\gamma}^{\mu} = P_{E}^{\mu} + P_{\gamma}^{\mu}$$

LHS

$$\begin{bmatrix}
\sqrt{p^2 + m^2 c^2} + E/c \\
P \cos \theta + E/c \cos \phi \\
P \sin \phi + E/c \sin \theta \\
0
\end{bmatrix} = \begin{bmatrix}
\sqrt{p_0^2 + m^2 c^2} + \hbar/\lambda_0 \\
P_0 \\
\hbar/\lambda_0 \\
0
\end{bmatrix}$$

RHS

taking the $x$-row:

$$P \cos \phi + E/c \cos \theta = P_0 \quad \cdots (1)$$

$$P \sin \phi + E/c \sin \theta = \hbar/\lambda_0 \quad \cdots (2)$$

$$\sqrt{p^2 + m^2 c^2} + E/c = \sqrt{p_0^2 + m^2 c^2} + \hbar/\lambda_0 \quad \cdots (3)$$

$$(1)^2 + (2)^2 
\Rightarrow p^2 + m^2 c^2 = \left( P_0 - \frac{E}{c} \cos \theta \right)^2 = \frac{p_0^2}{\lambda_0^2} - 2p_0 E \cos \theta + \frac{E^2}{c^2} \cos^2 \theta$$

$$+ \left( \frac{\hbar}{\lambda_0} - \frac{E}{c} \sin \theta \right)^2 = \frac{h^2}{\lambda_0^2} - \frac{2 \hbar E \sin \theta}{c \lambda_0} + \frac{E^2}{c^2} \sin^2 \theta$$

$$p^2 = P_0^2 + \frac{\hbar^2}{\lambda_0^2} - \frac{2p_0 E}{c} \sin \theta + \frac{2 \hbar E}{c} \sin \theta + \frac{E^2}{c^2}$$

Sub in to (3):

$$\Rightarrow \sqrt{p^2 + m^2 c^2} = \sqrt{P_0^2 + m^2 c^2} + \frac{\hbar}{\lambda_0} - \frac{E}{c}$$

$\cdots \cos \phi \cdots$
\[ E = \frac{\hbar}{\lambda_0} \sqrt{P_0^2 + m^2 c^2} \]

\[ = \frac{\hbar}{\lambda_0} \sqrt{\frac{P_0^2 + m^2 c^2}{c^2}} + \frac{\hbar}{\lambda_0} (1 - \sin \theta) - \frac{P_0}{c} \cos \theta \]

\[ h \sqrt{\frac{P_0^2 + m^2 c^2}{c^2}} + \frac{\hbar}{\lambda_0} (1 - \sin \theta) - \frac{P_0}{c} \cos \theta \]

\[ = h \sqrt{\frac{P_0^2 + m^2 c^2}{c^2}} + \frac{\hbar}{\lambda_0} (1 - \sin \theta) - \frac{P_0}{c} \cos \theta \]

\[ \Rightarrow \frac{\hbar E}{\lambda_0} (1 - \sin \theta) + \frac{\hbar^2 c^2}{\lambda_0} E - \frac{P_0}{c} \cos \theta E = \sqrt{P_0^2 + m^2 c^2} \]
*** 9 blank pages will be provided to each student and they can insert additional blank pages (also provided) if needed.