PHYS 7110 Final/Comprehensive Exam Mon. Dec 11, 2023

This exam is being graded with student identity anonymized. Please put your name and Unid on this page ONLY!!!

Name ____

Unid ________________________________

This exam has a strict time limit of two (2) hours. It will start at 1:00pm and finish at 3:00pm. There are three (3) problems.

Instructor's suggestions:

- Read every problem before you attempt to solve it.
- Do not spend more than 40 minutes on a problem until you have finished all the other problems.
- If you get stuck, move on to the next problem and come back to this one later.
- Integral Tables, Vector derivatives, Math Identities, Spherical Harmonics, Legendre polynomials and other special functions can be found in the "math" folder in CANVAS for this class.
- You can always use a symbolic math package, such as Maple, to evaluate integrals and do matrix multiplication.

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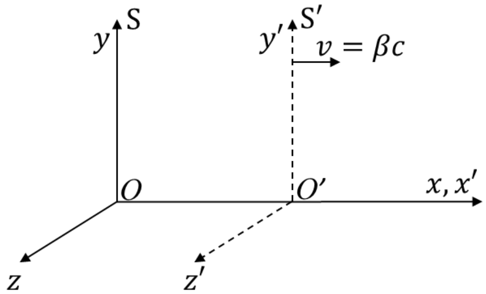
PHYS 7110 Fall 2022 Final/Comprehensive Exam Dec. 11, 2023

<u>Problem 1</u> [20 pts]

In this problem, we will show that the 4-gradient of an invariant function $f(x, y, z, t)$,



$$F_\alpha = \partial_\alpha f = \frac{\partial f}{\partial X^\alpha} = \begin{bmatrix} \frac{1}{c}\partial f/\partial t \\ \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{bmatrix}$$

is a covariant 4-vector. Follow the steps below. We take the usual situation where the moving frame S′ moves in the $+x$ direction relative to the lab frame S at velocity $v = \beta c$. The axes of the two systems are parallel as shown, and the origins O and O' coincide at $t = t' = 0$.

(a) [4 pts] We can treat the S frame space time coordinates t, x, y, z as functions of those in the S′ frame – i.e. $t = t(t', x', y', z')$, $x = x(t', x', y', z')$, $y = y(t', x', y', z')$, $z = z(t', x', y', z')$. Write down these four functions, you may include c, β, and $\gamma = 1/\sqrt{1-\beta^2}$. (These constitute the inverse Lorentz transformation).

(b) [6 pts] Now find all 4 components of $F'_\alpha = \partial_\alpha' f = \partial f/\partial X'^\alpha$— i.e. $\partial f/\partial t'$, $\partial f/\partial x'$, $\partial f/\partial y'$, and $\partial f/\partial z'$ by chain rule -- for example

$$\frac{\partial f}{\partial z'} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial z'} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial z'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z'} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial z'}$$

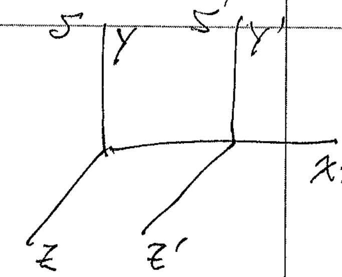
You must compute each $\partial X^\mu/\partial X'^\nu$ explicitly. Your answers should contain $\partial f/\partial t$, $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$.

Assuming F_α is a covariant 4-vector, we can also just apply Lorentz transformation $F'^\mu = L^\mu{}_\nu F^\nu$ (Einstein summation implied over repeated Greek indices). However, this is the transformation equation for a contravariant 4-vector

(c) [4 pts] You must first convert F_α from its covariant form to its contravaraint form F^α. This operation involves something like a multiplication of a 4x4 matrix (2nd order tensor) on the left of a column 4-vector. Write out your answer in the form

$$F^\alpha = \begin{bmatrix} ? \\ ? \\ ? \\ ? \end{bmatrix}$$

(d) [6 pts] Now apply the forward (from S to S' coordinates) Lorentz transformation to obtain F'^α. From these result, and NOT those from parts (a) and (b), find $\partial f/\partial t'$, $\partial f/\partial x'$, $\partial f/\partial y'$, and $\partial f/\partial z'$. Again your answers should contain $\partial f/\partial t$, $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$. Are these the same as what you got for part (b)?



(a) The inverse Lorentz transformation is given by

$$X^\alpha = (L^{-1})^\alpha{}_\sigma X'^\sigma$$

$$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix}$$

OR:

$$ct = \gamma(ct' + \beta x') \Rightarrow \boxed{t = t(t', x', y', z') = \gamma(t' + \tfrac{\beta}{c} x')}$$

$$x = \gamma(\beta c t' + x') \qquad \boxed{x = x(t', x', y', z') = \gamma(\beta c t' + x')}$$

$$y = y' \qquad \boxed{y = y(t', x', y', z') = y'}$$

$$z = z' \qquad \boxed{z = z(t', x', y', z') = z'}$$

(b) by defn:

$$F'_\alpha = \frac{\partial f'}{\partial X'^\alpha} = \frac{\partial f}{\partial X'^\alpha} = \begin{bmatrix} \frac{1}{c}\frac{\partial f}{\partial t'} \\ \frac{\partial f}{\partial x'} \\ \frac{\partial f}{\partial y'} \\ \frac{\partial f}{\partial z'} \end{bmatrix}$$

(f' ← invariant)

$$\frac{\partial f}{\partial t'} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial f}{\partial y}\cancel{\frac{\partial y}{\partial t'}}^{\,0} + \frac{\partial f}{\partial z}\cancel{\frac{\partial z}{\partial t'}}^{\,0}$$

$$= \gamma \frac{\partial f}{\partial t} + \gamma\beta c \frac{\partial f}{\partial x}$$

$$\boxed{\frac{1}{c}\frac{\partial f}{\partial t'} = \gamma\left(\frac{1}{c}\frac{\partial f}{\partial t} + \beta \frac{\partial f}{\partial x}\right)}$$

$$\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial t}\frac{\partial t}{\partial x'} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\cancel{\frac{\partial y}{\partial x'}}^{\,0} + \frac{\partial f}{\partial z}\cancel{\frac{\partial z}{\partial x'}}^{\,0}$$

$$\boxed{\frac{\partial f}{\partial x'} = \gamma\left(\beta \cdot \frac{1}{c}\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\right)}$$

$$\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial t}\cancel{\frac{\partial t}{\partial y'}}^{\,0} + \frac{\partial f}{\partial x}\cancel{\frac{\partial x}{\partial y'}}^{\,0} + \frac{\partial f}{\partial y}\cdot \underbrace{\frac{\partial y}{\partial y'}}_{=1} + \frac{\partial f}{\partial z}\cancel{\frac{\partial z}{\partial y'}}^{\,0}$$

(b) cont'd:

$$\Rightarrow \boxed{\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}}$$

$$\frac{\partial f}{\partial z'} = \frac{\partial f}{\partial t}\cancel{\frac{\partial t}{\partial z'}}^{0} + \frac{\partial f}{\partial x}\cancel{\frac{\partial x}{\partial z'}}^{0} + \frac{\partial f}{\partial y}\cancel{\frac{\partial y}{\partial z'}}^{0} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial z'}^{1}$$

$$\boxed{\frac{\partial f}{\partial z'} = \frac{\partial f}{\partial z}}$$

(c) We need to convert F_α to F^α in order to apply our standard Lorentz transformation

i.e $F'^\alpha = L^\alpha{}_\sigma F^\sigma$

And $F^\alpha = g^{\alpha\sigma} F_\sigma$ $\quad g^{\alpha\sigma}$ = metric tensor

$$\boxed{F^\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{c}\frac{\partial f}{\partial t} \\ \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{c}\frac{\partial f}{\partial t} \\ -\frac{\partial f}{\partial x} \\ -\frac{\partial f}{\partial y} \\ -\frac{\partial f}{\partial z} \end{bmatrix}}$$

(d)
$$F'^\alpha = \begin{bmatrix} \frac{1}{c}\frac{\partial f}{\partial t'} \\ -\frac{\partial f}{\partial x'} \\ -\frac{\partial f}{\partial y'} \\ -\frac{\partial f}{\partial z'} \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{c}\frac{\partial f}{\partial t} \\ -\frac{\partial f}{\partial x} \\ -\frac{\partial f}{\partial y} \\ -\frac{\partial f}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} \gamma\left(\frac{1}{c}\frac{\partial f}{\partial t} + \beta\frac{\partial f}{\partial x}\right) \\ -\gamma\left(\beta\frac{1}{c}\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\right) \\ -\frac{\partial f}{\partial y} \\ -\frac{\partial f}{\partial z} \end{bmatrix}$$

... cont'd

(d) We thus have

$$\boxed{\frac{1}{c}\frac{\partial f}{\partial t'} = \gamma\left(\frac{1}{c}\frac{\partial f}{\partial t} + \beta\frac{\partial f}{\partial x}\right)}$$ ... same as in (b) ✓

$$\boxed{\frac{\partial f}{\partial x'} = \gamma\left(\beta\frac{1}{c}\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\right)}$$ ... same as in (b) ✓

$$\boxed{\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}}$$ ✓

Same as (b)

$$\boxed{\frac{\partial f}{\partial z'} = \frac{\partial f}{\partial z}}$$ ✓

These results assumed $\partial_\mu f$ is a covariant 4-vector

and gave the same result as chain rule

$\Rightarrow$ $\boxed{\partial_\mu f \text{ is a covariant 4-vector}}$

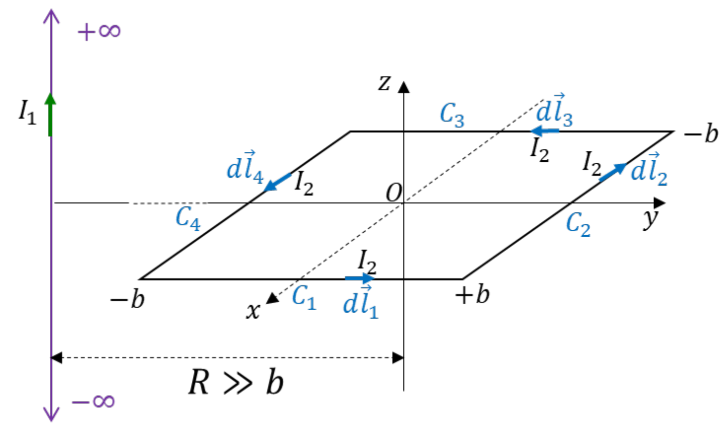
assuming f is an invariant function !

Problem 2 [20 pts]

A square loop carries a current I_2 that circulates in the counter-clockwise sense as seen from above as shown. The loop is centered on the origin and sits in the xy-plane. It has sides of length $2b$.

An infinite wire lies parallel to the z-axis. It is offset in the negative y direction from the origin by a distance R (i.e. it sits at $x = 0$, $y = -R$, $R \gg b$).

This wire carries a current I_1 in the +z direction. Treating the loop as a point magnetic dipole, find the force and torque exerted by the wire on the loop, following the steps below.



We break up the loop into four segments, C_1, C_2, C_3, and C_4.

(a) [3 pts] Write down the line element $d\vec{l}_1$ and its location $\vec{r}_1$ on segment C_1, in Cartesian coordinates, x, y, z, their differentials dx, dy, dz, and in Cartesian components – i.e. as a linear combination of $\hat{x}$, $\hat{y}$, $\hat{z}$.

(b) [3 pts] Integrate the appropriate combination of $d\vec{l}_1$ and $\vec{r}_1$ over C_1 to find $\vec{m}_1$, the contribution of C_1 to the total magnetic (dipole) moment $\vec{m}$.

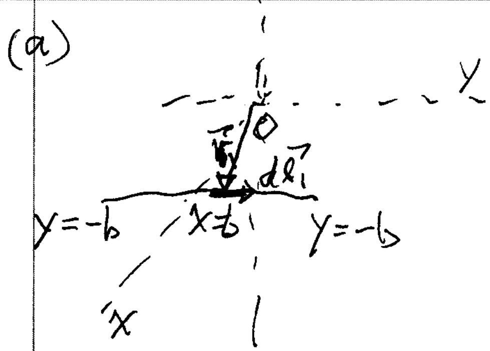
(c) [2 pts] Use the symmetry of the system to find $\vec{m}$ from $\vec{m}_1$.

(d) [4 pts] Write down the magnetic field $\vec{B}(\vec{r})$ generated by current I_1 in Cartesian coordinates, x, y, z, and in Cartesian components – i.e. as a linear combination of $\hat{x}$, $\hat{y}$, $\hat{z}$. Remember the infinite wire lies parallel to the z-axis and is located at $x = 0$, $y = -R$, where $R \gg b$.

(e) [4 pts] From the results of (c) and (d) find the torque $\vec{N}$ exerted by the magnetic field generated by the infinite wire on the current loop, in the dipole approximation, in Cartesian components – i.e. as a linear combination of $\hat{x}$, $\hat{y}$, $\hat{z}$.

(f) [4 pts] From the results of (c) and (d) find the force $\vec{F}$ exerted by the magnetic field generated by the infinite wire on the current loop, in the dipole approximation, in Cartesian components – i.e. as a linear combination of $\hat{x}$, $\hat{y}$, $\hat{z}$.

(a)



$$\boxed{\vec{r}_1 = b\hat{x} + y\hat{y} + \cancel{0}\hat{z}}$$

$-b < y < b$ in xy plane

$$\boxed{d\vec{l_1} = dy\,\hat{y}}$$

(b) $\vec{m}_1 = \frac{1}{2}\int_{C_1} \vec{r}_1 \times dq_1 \vec{v}_1 = \frac{1}{2}\int_{C_1} \vec{r}_1 \times I_1 d\vec{l}_1$

$\vec{r}_1 = b\hat{x} + y\hat{y} \quad d\vec{l}_1 = dy\,\hat{y}$

$\hat{x}\times\hat{y} = \hat{z}, \; \hat{y}\times\hat{y} = 0$

$\frac{1}{2}\vec{r}_1 \times I_1 d\vec{l}_1 = \frac{I_1}{2}(b\hat{x} + y\hat{y}) \times dy\,\hat{y} = \frac{I_1 b}{2} dy\,\hat{z}$

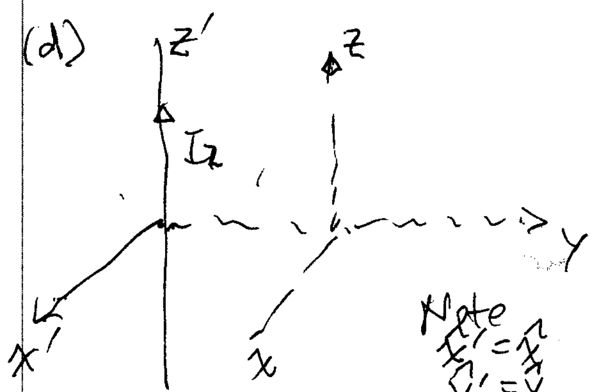
$\vec{m}_1 = \int_{-b}^{b} \frac{I_1 b}{2} dy\,\hat{z} \quad \frac{I_1 b}{2}\hat{z} \cdot \int_{-b}^{b} dy = \frac{I_1 b}{2}\hat{z} \cdot 2b = b^2 I_1 \hat{z}$

(c) By symmetry, all 4 segments should contribute the same i.e. $\vec{m}_2 = \vec{m}_3 = \vec{m}_4 = \vec{m}_1$

$\Rightarrow \vec{m} = 4\vec{m}_1 = \boxed{4b^2 I_1 \hat{z}}$

✲ NOTE we could have guessed $\vec{m} = I_1 \vec{a}$ where $\vec{a} = (2b)^2\hat{z}$ is the area vector of C

(d)



Note
$\hat{x}' = \hat{x}$
$\hat{y}' = \hat{y}$
$\hat{z}' = \hat{z}$

Let's define a cylindrical coordinate system centered @ $O' = -R\hat{y}$

$\Rightarrow x' = s\cos\phi$
$y' = s\sin\phi$
$z' = z$

$x = x'$
$y = y' - R$
$z = z'$

$\vec{B} = \frac{\mu_0 I_2}{2\pi s}\hat{s} = \frac{\mu_0 I_2}{2\pi} \cdot \frac{s\cos\phi\,\hat{x} + s\sin\phi\,\hat{y}}{s^2} = \frac{\mu_0 I_2}{2\pi} \frac{x'\hat{x} + y'\hat{y}}{[x'^2 + y'^2]}$

Known result from class

(d) cont'd: $x' = x \quad y' = y + R$

double check: O' is at $y' = 0$
$\Rightarrow y = y' - R = -R$ ✓

$$\Rightarrow \boxed{\vec{B}(\vec{r}) = \mu_0 I_2 \frac{x\hat{x} + (y+R)\hat{y}}{[x^2 + (y+R)^2]}}$$

(e) In the point dipole approximation; the torque on $\vec{m}$ is

$$\vec{N} = \vec{m} \times \vec{B}(\underset{\uparrow}{0})$$

location of dipole

i.e. we want to find $\vec{B}$ at O

$$\vec{B}(0) = \mu_0 I_2 \frac{\cancel{(0)\hat{x}} + (R)\hat{y}}{[\cancel{(0)}^2 + (0+R)^2]} = \frac{\mu_0 I_2 \hat{y}}{R}$$

$\hat{z} \times \hat{y} = -\hat{x}$

$$\boxed{\vec{N}} = \underbrace{4b^2 I_1 \hat{z}}_{\vec{m}} \times \frac{\mu_0 I_2 \hat{y}}{R} = \boxed{\frac{4\mu_0 b^2 I_1 I_2}{R}\hat{x}}$$

(f) One could use one of two equivalent expressions for the force $\vec{F}$

$$\vec{F} = \left[\vec{\nabla}(\vec{m} \cdot \vec{B})\right]_{\vec{r}=0}$$

$\vec{m}$ is a constant vector

Vector identity: $\vec{\nabla}(\vec{m} \cdot \vec{B}) = \vec{m} \times \underbrace{(\vec{\nabla} \times \vec{B})} + \vec{B} \times \cancel{(\vec{\nabla} \times \vec{m})}^{0} + (\vec{m} \cdot \vec{\nabla})\vec{B} + \cancel{(\vec{B} \cdot \vec{\nabla})\vec{m}}^{0}$

$$\left(\vec{\nabla} \times \vec{B}\right)\Big]_0 = 0 = \mu_0 \vec{J}(0) \text{ here:}$$

it is an external field!

$$\vec{\nabla}(\vec{m} \cdot \vec{B})\Big|_0 = (\vec{m} \cdot \vec{\nabla})\vec{B}\Big|_0$$

First:

$$\vec{m}\cdot\vec{B} = 4b^2 I_1 \hat{z}\cdot \mu_0 I_2 \frac{x\hat{x} + (y+R)\hat{y}}{[x^2+(y+R)^2]}$$

$$\hat{z}\cdot\hat{x} = 0 \qquad \hat{z}\cdot\hat{y} = 0$$

$$\Rightarrow \vec{m}\cdot\vec{B} = 0$$

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})\Big|_0 = 0$$

OR: 2nd form

$$\vec{F} = \Big(\cancel{m_x}^{\,0}\frac{\partial}{\partial x} \quad \cancel{m_y}^{\,0}\frac{\partial}{\partial y} + \underset{4I_1 b^2}{\underset{\parallel}{m_z}}\frac{\partial}{\partial z}\Big)\underbrace{\mu_0 I_2 \frac{x\hat{x} + (y+R)\hat{y}}{[x^2+(y+R)^2]}}_{\vec{B}}\Bigg|_0$$

Note $\frac{\partial}{\partial z}\vec{B} = 0$ everywhere

$$\Rightarrow \vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B}\Big|_0 = 0$$

Same result!

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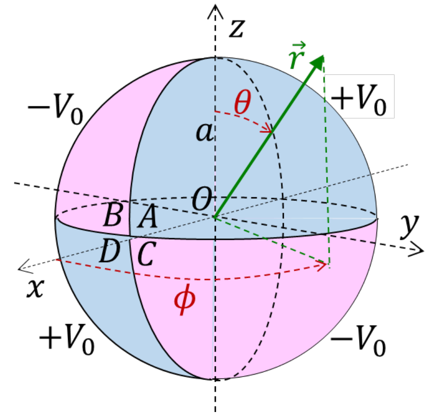
Problem 3 [20 pts]

A solid conducting sphere has radius a, and is centered on the origin. It is divided into four quadrants about the x-axis as shown, such that each piece is held at an alternating potential of $\varphi = \pm V_0$:

$$\varphi(a,\theta,\phi) = \begin{cases} +V_0 & \text{region } A \quad z>0 \quad y>0 \\ -V_0 & \text{region } B \quad z>0 \quad y<0 \\ -V_0 & \text{region } C \quad z<0 \quad y>0 \\ +V_0 & \text{region } D \quad z<0 \quad y<0 \end{cases}$$



The sphere sits in a space that is empty for $r > a$. We will be investigating the electrostatic potential $\varphi(r,\theta,\phi)$ in this (outside) region in spherical coordinates.

(a) [4 pts] Find the limits in θ and ϕ for the four regions A, B, C, and D, in the form, for example, a hypothetical region F:

region F: $\pi/4 < \theta < 3\pi/4$, and $-\pi/3 < \phi < -\pi/6$

(b) [2 pts.] Write down the most general solution $\varphi(r,\theta,\phi)$ to the Laplace Equation, when solved by separation of variables in spherical coordinates r, θ, ϕ, with spherical boundary conditions. This should be an infinite series summing over two indices, l and m. As we have done in class, use the coefficients A_l for the non-negative (zero or positive) powers of r, and B_l for the negative powers of r.

(c) [4 pts] Apply the implicit boundary condition that the potential $\varphi \to 0$ as $r \to \infty$. This should eliminate half of the coefficients (i.e. they are all zero for all values of l and m.). Indicate which coefficients vanish from this boundary condition and write the new, now restricted general solution for $r > a$.

(d) [10 pts] Now apply the stated boundary condition at $r = a$. Solve for the coefficients for $l = 2$ and all allowed values of m.

Spherical Harmonics:

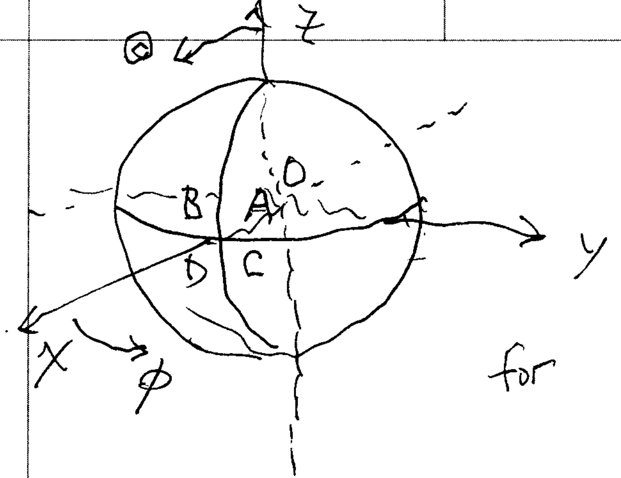
$l = 0$: $\quad Y_{00} = \dfrac{1}{\sqrt{4\pi}}$

$l = 1$:
$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$$
$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$l = 2$:
$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{2i\phi}$$
$$Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}$$
$$Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$$

Remember for negative m, use

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y^*_{lm}(\theta,\phi)$$



(a) for $Z>0$:
i.e regions A, B
we have $0<\Theta<\pi/2$
and $\pi/2<\Theta<\pi$ for C D $Z<0$

for regions A, C : $y>0 \Rightarrow 0<\phi<\pi$
B D $y<0$ $-\pi<\phi<0$

$$\boxed{\begin{array}{llll} \text{Region A :} & 0<\Theta<\frac{\pi}{2} & 0<\phi<\pi & \varphi=+V_0 \\ \text{B} & 0<\Theta<\frac{\pi}{2} & -\pi<\phi<0 & \varphi=-V_0 \\ \text{C} & \frac{\pi}{2}<\Theta<\pi & 0<\phi<\pi & \varphi=-V_0 \\ \text{D} & \frac{\pi}{2}<\Theta<\pi & -\pi<\phi<0 & \varphi=+V_0 \end{array}}$$

(b) $$\boxed{\varphi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left[A_{lm}r^{l}+\frac{B_{lm}}{r^{l+1}}\right]Y_{lm}(\theta,\phi)}$$

is the most general solution in spherical coordinates on spherical boundary conditions.

(c) $$\varphi \xrightarrow[r\to\infty]{} = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left[\underbrace{A_{lm}(r)^{l}}_{\infty}+\frac{B_{lm}}{(r)^{l+1}}\right]Y_{lm}(\theta,\phi)$$

So $A_{lm}r^{l}\to\infty$ for $l\geq 1$ as $r\to\infty$

$\Rightarrow A_{lm}=0$ for $l\geq 1$ in order $\varphi\to 0$

Also : $A_{00}r^{0}=A_{00}\to A_{00}$ as $r\to\infty$

$\Rightarrow$ we also require $A_{00}=0$

$$\Rightarrow \boxed{\varphi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-\infty}^{\infty}\frac{B_{lm}}{r^{l+1}}Y_{lm}(\theta,\phi) \quad \text{after requiring } \varphi\xrightarrow[r\to\infty]{}0}$$

(d) We take advantage of the orthogonality/normality condition of $Y_{lm}(\theta,\phi)$

i.e: $\int d\Omega\, Y_{lm}^*(\theta,\phi)\, Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$

so taking (using l', m') Φ at $r=a$

$$\Phi(a,\theta,\phi) = \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{+l'} \frac{B_{l'm'}}{a^{l'+1}} Y_{l'm'}(\theta,\phi).$$

$$\int d\Omega\, Y_{lm}^*(\theta,\phi)\,\Phi(a,\theta,\phi) = \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{+l'} \frac{B_{l'm'}}{a^{l'+1}} \int d\Omega\, Y_{lm}^*(\theta,\phi)\, Y_{l'm'}(\theta,\phi)$$

This is NOT necessary

$$= \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{+l'} \frac{B_{l'm'}}{a^{l'+1}}\delta_{ll'}\delta_{mm'} = \frac{B_{lm}}{a^{l+1}}$$

$$\Rightarrow B_{lm} = a^{l+1}\int d\Omega\, Y_{lm}^*(\theta,\phi)\,\Phi(a,\theta,\phi)$$

We are interested only in $l=2$

We know, however, that $Y_{l,-m}(\theta,\phi) = (-1)^m Y_{lm}^*(\theta,\phi)$

$$\Rightarrow B_{l-m} = (-1)^m B_{lm}$$

$l=2,\ m=0$

$$Y_{20}^*(\theta,\phi) = \left[\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\right]^*$$

$$= \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$B_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\,a^3 V_0 \Big\{ \int_0^{\cancel{+\pi}\ \pi} d\phi \int_0^{\pi/2} d\theta \sin\theta\left(3\cos^2\theta - \frac{1}{2}\right) \qquad A$$

$$\int_{\cancel{-\pi}\ \pi}^{\cancel{0}\ 2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta\left(3\cos^2\theta - \frac{1}{2}\right) \qquad B$$

$$- \int_0^{\cancel{+\pi}\ 2\pi} d\phi \int_{\pi/2}^{\pi} d\theta \sin\theta\left(3\cos^2\theta - \frac{1}{2}\right) \qquad C$$

$$- \int_{\cancel{-\pi}\ \pi}^{\cancel{0}\ \pi} d\phi \int_{\pi/2}^{\pi} d\theta \sin\theta\left(3\cos^2\theta - \frac{1}{2}\right)\Big\} \qquad D.$$

$$\int (3\cos^2\theta - 1)\sin\theta\, d\theta = -\int (3u^2-1)\,du = -(u^3-u)$$

$$\int_0^{\pi/2} (3\cos^2\theta - 1)\sin\theta\, d\theta = -\left(u^3-u\right)_1^0 = -\left[\cancel{0-0-1+1}\right] = 0$$

$$\int_{\pi/2}^{\pi} (3\cos^2\theta - 1)\sin\theta\, d\theta \quad -\left[u^3-u\right]_0^{-1} = -\left[\cancel{-1+1-0+0}\right] = 0$$

$$B_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}}\, a^3 \qquad [0 \quad 0-0-0] = 0$$

$$\boxed{B_{20} = 0}$$

$l = 2,\ m = 1$

$$Y_{21}^*(\theta,\phi) = \left[-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}\right]^*$$

$$= -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{-i\phi}$$

$$B_{21} = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\, a^3 V_0 \left\{ \int_0^{+\pi} e^{-i\phi}d\phi \int_0^{\pi/2} \sin^2\theta\cos\theta\, d\theta \right. \qquad A$$

$$- \int_{-\pi}^{0} e^{-i\phi}d\phi \int_0^{\pi/2} \sin^2\theta\cos\theta\, d\theta \qquad B$$

$$- \int_{0}^{\pi} e^{-i\phi}d\phi \int_{\pi/2}^{\pi} \sin^2\theta\cos\theta\, d\theta \qquad C$$

$$\left. + \int_{-\pi}^{0} e^{-i\phi}d\phi \int_{\pi/2}^{\pi} \sin^2\theta\cos\theta\, d\theta \right\} \qquad D$$

$$\int e^{-i\phi}d\phi = \frac{1}{-i}e^{-i\phi} = i e^{-i\phi}$$

$$\int_0^{\pi} e^{-i\phi}d\phi = \left(i e^{-i\phi}\right)_0^{\pi} = i[-1-1] = -2i$$

$$\int_{-\pi}^{0} e^{-i\phi} = i[1-(-1)] = 2i$$

$v = \sin\theta \qquad dv = \cos\theta\, d\theta$

$$\int \sin^2\theta\cos\theta\, d\theta = \int v^2 dv = \frac{1}{3}v^3$$

$$\int_0^{\pi/2} \sin^2\theta\cos\theta\, d\theta = \frac{1}{3}\left[v^3\right]_0^1 = \frac{1}{3}$$

$$\int_{\pi/2}^{\pi} \sin^2\theta\cos\theta\, d\theta = \frac{1}{3}\left[v^3\right]_1^0 = -\frac{1}{3}$$

$$B_{21} = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}\, a^3 V_0 \left\{ (-2i)\left(\frac{1}{3}\right) - (+2i)\left(\frac{1}{3}\right) - (-2i)\left(-\frac{1}{3}\right) + (2i)\left(-\frac{1}{3}\right) \right\}$$

$$\boxed{B_{21} = -\frac{1}{2}\cdot -\frac{8i}{3}\sqrt{\frac{15}{2\pi}}\, a^3 V_0 = 2i\sqrt{\frac{10}{3\pi}}\, a^3 V_0}$$

$$B_{2-1} = (-1)^1 B_{21}^* \rightarrow \boxed{B_{2-1} = 2i\sqrt{\frac{10}{3\pi}}\, a^3 V_0} \qquad (-1)(i)^* = 1$$

$l=2 \quad m=2$

$$Y_{22}^*(\theta,\phi) = \left[\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}\right]^* = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{-2i\phi}$$

$$\Rightarrow B_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\, a^3 V_0 \left\{\int_0^{+\pi} e^{-2i\phi}d\phi \int_0^{\pi/2}\sin^3\theta\, d\theta\right.$$

$$-\int_{-\pi}^{0} e^{-2i\phi}d\phi \int_0^{\pi/2}\sin^3\theta\, d\theta$$

$$-\int_0^{+\pi} e^{-2i\phi}d\phi \int_{\pi/2}^{\pi}\sin^3\theta\, d\theta$$

$$+\int_{-\pi}^{0} e^{-2i\phi}d\phi \int_{\pi/2}^{\pi}\sin^3\theta\, d\theta$$

Note $\int_0^{\pi} e^{-2i\phi} = \int_{-\pi}^{0} e^{-2i\phi} = 0$

because both are integrals over full period for $e^{-2i\phi}$

$$\Rightarrow \boxed{B_{22} = 0} \qquad \boxed{B_{2-2} = (-1)^2 B_{22}^* = 0}$$