PHYS 7220 Quantum Theory I (Final/Comprehensive Exam), 12/13/2023

This exam is being graded with student identity anonymized. Please put your name and Unid on this page ONLY!!!

Name: ______________________________

Unid: ___________

- This exam has a strict time limit of two (2) hours. It will start on Wednesday December 13 at 1:00 pm and conclude at 3:00 pm.
- The exam is worth 100 points. There are three (3) problems, Problem 1 is worth 30 points, Problem 2 is worth 30 points, and Problem 3 is worth 40 points.

Instructor's suggestions:

- Read every problem carefully before attempting to solve it.
- Keep track of time. Do not spend an excessive amount of time on one or two problems at the expense of other problems.
- If you get stuck, move on to the next problem and come back to this one later.

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Problem 1. (30 points) An electron is incident on an infinitely thin sheet of surface charge $\sigma < 0$ (measured in Coulombs per square meter) perpendicularly to the sheet. The energy of the electron is such that, according to classical mechanics, the electron reflects back from the sheet with the minimum distance from the electron to the sheet being a. By following the steps outlined below, find the probability of the electron tunneling through the sheet. Ignore possible electron collisions with individual particles of the sheet. Assume that the probability is small and that the electron charge $-e$ and mass m are known.

a) Determine the force acting on the electron on both sides of the sheet. (5 points)

b) Determine potential energy $U(x)$ of the electron on both sides of the sheet. (5 points)

c) Determine the total energy of the electron. (5 points)

d) Identify an appropriate integral expression for the probability of tunneling. (5 points)

e) Calculate the integral from part c). (5 points)

f) Check the dimension of your result and comment on it. (5 points)

Problem 2. (30 points) Consider a one-dimensional harmonic oscillator described by the Hamiltonian,

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2},$$

where m is the mass of the oscillator and ω is its frequency. By following the steps outlined below, find the correlation function $K(t) = \langle n|x(t)p(0)|n\rangle$ for the n^{th} state of the harmonic oscillator. Here $x(t)$ is the position operator in the Heisenberg picture, and $p(t)$ is the momentum operator in the Heisenberg picture.

a) Express the time-independent Schrödinger operators x and p via creation and annihilation operators. (7 points)
b) Write the time-dependent Heisenberg operators $x(t)$ and $p(t)$. (8 points)
c) Use the obtained Heisenberg operators $x(t)$ and $p(t)$ to find the correlation function $K(t) = \langle n|x(t)p(0)|n\rangle$. (12 points)
d) Check the dimension of your result and comment on it. (3 points)

Problem 3. (40 points) Interaction of two spin-1/2 particles is described by the following Hamiltonian,

$$H = \frac{2J_1}{\hbar^2}\left(s_{1x}s_{2x} - s_{1y}s_{2y}\right) + \frac{4J_2}{\hbar^2}s_{1z}s_{2z}.$$

Here $\mathbf{s}_1 = (s_{1x},\, s_{1y},\, s_{1z})$ and $\mathbf{s}_2 = (s_{2x},\, s_{2y},\, s_{2z})$ are the spin operators of the first particle and the second particle, respectively, and J_1 and J_2 are known positive constants. By following the steps outlined below, find the eigenvalues and eigenstates of this Hamiltonian.

a) Write each term of the Hamiltonian as a 4×4 matrix representing a direct product of 2×2 matrices of individual spin operators. (10 points)

b) Write the full Hamiltonian as a 4×4 matrix. (5 points)

c) Find all eigenvalues of the full Hamiltonian of part b). (10 points)

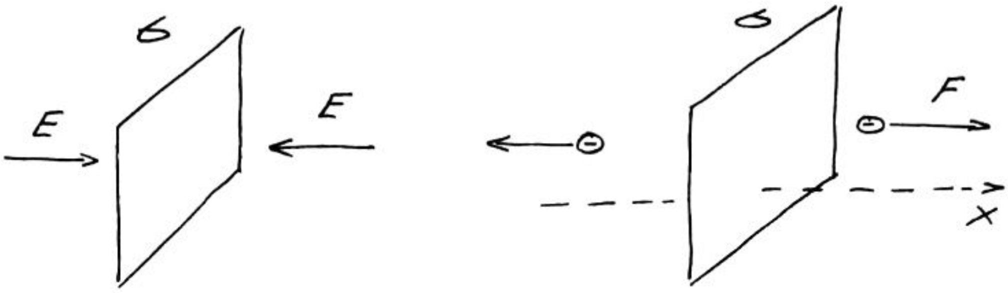
d) Identify all eigenstates of the full Hamiltonian. (10 points)

e) For what relation between J_1 and J_2 is the ground state non-degenerate? (5 points)

Problem 1

Electric field of the sheet is $E = \frac{\sigma}{2\varepsilon_0}$

a)



Force acting on electron

$$F(x) = \begin{cases} F, & x>0 \\ -F, & x<0 \end{cases} \qquad F = \frac{e\sigma}{2\varepsilon_0}$$

Potential energy $U(x)$ is found from the condition

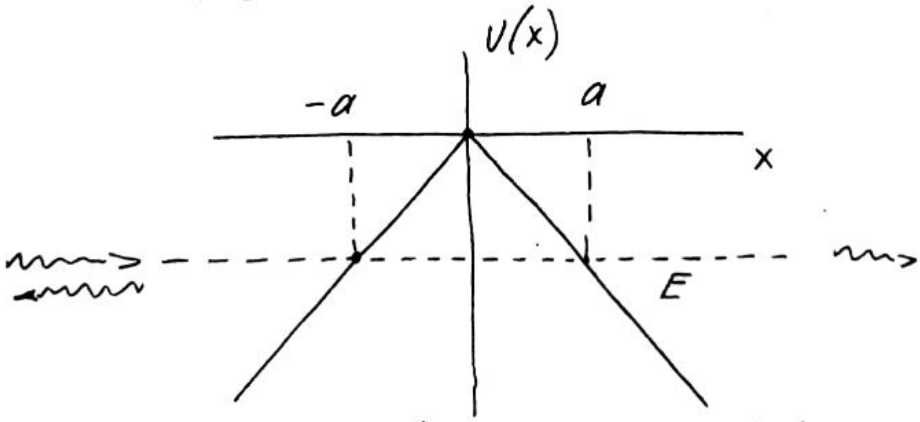
b)

$$F(x) = -\frac{\partial U(x)}{\partial x}$$

$$U(x) = \begin{cases} -Fx, & x>0 \\ Fx, & x>0 \end{cases}$$

or, simply, $U(x) = -F|x|$

c)



The minimum distance a determines classical turning points $\pm a$:

$$E = -Fa$$

The probability of tunneling through the barrier $U(x)$, according to WKB approximation:

$$T = \exp\left(-\frac{2}{\hbar}\int_{-a}^{a} |p(x)|dx\right)$$

where $|p(x)| = \sqrt{2m[U(x)-E]}$

$$= \sqrt{2mF(a-|x|)}$$

d) $$T = \exp\left(-\frac{4}{\hbar}\int_0^a |p(x)|dx\right) =$$

$$= \exp\left(-\frac{4}{\hbar}\sqrt{2mF}\underbrace{\int_0^a \sqrt{a-x}\,dx}_{\frac{2}{3}a^{3/2}}\right)$$

$$= \exp\left(-\frac{8a^{3/2}\sqrt{2mF}}{3\hbar}\right)$$

e) Dimension check

$$\sim \underbrace{\sqrt{\frac{ma^2}{\hbar^2}}}_{\text{inverse energy}}\ \underbrace{\sqrt{Fa}}_{\text{energy}}$$

Dimensionless – OK

Problem 2

a) In the representation of creation a^+ and annihilation operators

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} = \hbar\omega\left(a^+a + \frac{1}{2}\right)$$

where $x = \sqrt{\frac{\hbar}{2m\omega}}(a^+ + a)$

$$p = i\sqrt{\frac{m\hbar\omega}{2}}(a^+ - a)$$

b) In the Heisenberg picture, operators $a^+(t)$ and $a(t)$ depend on time

$$a^+(t) = \underbrace{a^+}_{} e^{i\omega t} \qquad a(t) = \underbrace{a}_{} e^{-i\omega t}$$

Schrödinger operators

This follows, e.g. from

$$\frac{da(t)}{dt} = \frac{i}{\hbar}[H, a(t)] = \frac{i}{\hbar} \times \hbar\omega \, [a^+(t)a(t), a(t)]$$

$$= i\omega \left(a^+(t)a(t) - a(t)a^+(t) \right) a(t)$$

$$= -i\omega a(t)$$

Thus $\quad x(t) = \sqrt{\frac{\hbar}{2m\omega}} \left(a^+ e^{i\omega t} + a e^{-i\omega t} \right)$

$$p(0) = i\sqrt{\frac{m\hbar\omega}{2}} \left(a^+ - a \right)$$

c) $K(t) \equiv \langle n | x(t) p(0) | n \rangle = i \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\frac{m\hbar\omega}{2}} \times$

$$\times \langle n | \left(a^+ e^{i\omega t} + a e^{-i\omega t} \right) \left(a^+ - a \right) | n \rangle$$

$$= i \frac{\hbar}{2} \left\{ \underbrace{\langle n | a a^+ | n \rangle}_{n+1} e^{-i\omega t} - \underbrace{\langle n | a^+ a | n \rangle}_{n} e^{i\omega t} \right\}$$

(Remember that $a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$)

Therefore,

$$K(t) = \frac{i\hbar}{2}\left\{ (n+1)e^{-i\omega t} - n e^{i\omega t} \right\}$$

$$= \hbar n \sin\omega t + \frac{i\hbar}{2} e^{-i\omega t}$$

d) Dimension - ok: product of x and p has dimension of $\hbar$

Problem 3

a) In the direct-product representation of the Hilbert spaces of the two spins:

$$H = \frac{1}{2}J_1\left(\sigma_{1x}\otimes\sigma_{2x} - \sigma_{1y}\otimes\sigma_{2y}\right) + J_2\,\sigma_{1z}\otimes\sigma_{2z}$$

$$\sigma_{1x}\otimes\sigma_{2x} = \begin{pmatrix} 0 & \sigma_{2x} \\ \sigma_{2x} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_{1y}\otimes\sigma_{1y} = \begin{pmatrix} 0 & -i\sigma_{2y} \\ i\sigma_{2y} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_{1z}\otimes\sigma_{2z} = \begin{pmatrix} \sigma_{2z} & 0 \\ 0 & -\sigma_{2z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b) All together:

$$H = \begin{pmatrix} J_2 & 0 & 0 & J_1 \\ 0 & -J_2 & 0 & 0 \\ 0 & 0 & -J_2 & 0 \\ J_1 & 0 & 0 & J_2 \end{pmatrix}$$

This Hamiltonian acts on states represented by vectors

$$\begin{pmatrix} C_{++} \\ C_{+-} \\ C_{-+} \\ C_{--} \end{pmatrix} \qquad |\alpha\rangle = C_{++}|++\rangle + C_{+-}|+-\rangle + C_{-+}|-+\rangle + C_{--}|--\rangle$$

where the first subscript in $C_{\alpha\beta}$ indicates z-component of the first spin and the second subscript β indicates z-component of the second spin.

c) From the form of the Hamiltonian, it follows that states $|+-\rangle$ and $|-+\rangle$ are degenerate eigenstates of H with energy $\boxed{E = -J_2}$

The other two states are found from 2×2 system of equations

$$J_2 C_{++} + J_1 C_{--} = E C_{++}$$

$$J_1 C_{++} + J_2 C_{--} = E C_{--}$$

$$(J_2 - E)^2 - J_1^2 = 0 \quad \Rightarrow \quad \boxed{\begin{aligned} E_a &= J_2 + J_1 \\ E_b &= J_2 - J_1 \end{aligned}}$$

d) for E_a state, we have from, e.g. the first of the two linear equations

$$J_2 C_{++} + J_1 C_{--} = (J_2 + J_1) C_{++}$$

or $C_{++} = C_{--} \Rightarrow C_{++} = C_{--} = \frac{1}{\sqrt{2}}$ (upon normalization

Similarly, we ged for E_6 state:

$$C_{++} = \frac{1}{\sqrt{2}} \qquad C_{--} = -\frac{1}{\sqrt{2}}$$

Thus, eigestates/eigenvalues are

e) Ground state is non-degenerate if

$J_2 - J_1 < -J_2$

Thus $2J_2 < J_1$

1. Any two orthogonal combinations of stades $|+-\rangle$ and $|-+\rangle$ with energy
$E = -J_2$

2. Symmetric combination $\frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)$ with energy $E = J_2 + J_1$

3. Anti-symmetric combination $\frac{1}{\sqrt{2}}(|++\rangle - |--\rangle)$ with energy $E = J_2 - J_1$

In the Basis $|S, S_z\rangle$ of total spin/z-component of total spin, we have

1. Any two orthogonal combinations of stades $|1,0\rangle$ and $|0,0\rangle$

2. Symmetric combination $\frac{1}{\sqrt{2}}(|1,+1\rangle + |1,-1\rangle)$ with energy $E = J_2 + J_1$

3. Anti-symmetric combination $\frac{1}{\sqrt{2}}(|1,+1\rangle - |1,-1\rangle)$ with energy $E = J_2 - J_1$