The bottle that drains at a steady speed

Poke a hole near the bottom of a water bottle and the jet starts strong, then gets weaker as the bottle empties. In the 1600s, the French physicist Edme Mariotte found a simple way to make the jet stay *exactly* the same until almost the end. Next month's topic is moving fluids. This problem gets you started.

Bring whatever you get to the next meeting: a full answer, part of an answer, a good question, or a bottle. Everyone should try Part A. Parts B and C are harder.

Part A: How fast is the jet?

An open bottle has water to a height h above a small hole. Ignore stickiness (viscosity).

(a) In a short time, a small mass m of water squirts out of the hole. The net effect is the same as if m had vanished from the *top* of the water and reappeared, moving, at the hole. Use energy to find the jet's speed v in terms of g and h. Does this formula look familiar?

(b) Find v for $h = 20$ cm.

(c) The bottle has cross-section area A and the hole has area a. Write dh/dt in terms of v, a, and A. Solve it to find how long the bottle takes to drain from height h_0.

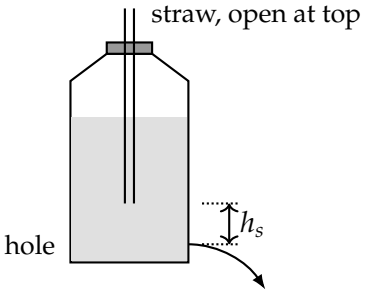
(d) Show that the bottle takes *twice* as long to drain as it would if the jet kept its starting speed the whole time.

Calculus toolbox for (c)

Your equation looks like $dh/dt = -k\sqrt{h}$. Separate the variables: $h^{-1/2}\,dh = -k\,dt$. Use $\int h^{-1/2}\,dh = 2\sqrt{h}$, then solve for the time when h reaches 0. *No calculus?* Make a table with 10-second time steps. In each step, the water level drops by about $(a/A)\,v\,\Delta t$, using v at the start of the step. Then plot h versus t.

Part B: Mariotte's trick

Now seal the bottle with a cap. Push a straw down through a tight hole in the cap, so the straw's bottom end is underwater, a height h_s above the side hole. The top of the straw is open to the air.



(e) As water squirts out, bubbles of air come in through the bottom of the straw. What is the pressure at the bottom of the straw? What about in the air space at the top of the bottle? (It isn't 1 atm!)

(f) Explain why the jet speed stays the same while the water level falls, as long as the water is above the bottom of the straw. What is the speed?

(g) What happens once the water level drops below the bottom of the straw?

Part C: Build one

(h) Make one from a plastic bottle, a straw, and some clay or hot glue to seal the cap. Measure how far the jet lands from the bottle, and predict it first. Does moving the straw up or down change the jet the way you expect?

Extra challenge

Your formula in Part A ignores stickiness. For your homemade bottle, is that a good approximation? How could you test it?

Numbers: $g = 9.8\ \text{m/s}^2$ density of water $= 1000\ \text{kg/m}^3$ $1\ \text{atm} \approx 1.0 \times 10^5\ \text{Pa}$