The Periodic Standing Wave Approximation: Adapted coordinates and spectral methods

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Abstract

The periodic standing wave method for the binary inspiral of black holes and neutron stars computes exact numerical solutions for periodic standing wave spacetimes and then extracts approximate solutions of the physical problem, with outgoing waves. The method requires solution of a boundary value problem with a mixed (hyperbolic and elliptic) character. We present here an approach based on a coordinate system adapted to the geometry of the problem and on a spectral method. Results presented for nonlinear model problems suggest that the advantages of the adapted coordinates and spectral method might warrant the extra complexity they entail.

I. INTRODUCTION

A. Background

The intro must be rewritten

The detection and interpretation of gravitational wave signals, from inspiralling black holes or neutron stars, requires a solution of Einstein's equations for the late stages of the inspiral[1, 2]. Much effort is going into the development of computer codes that will evolve solutions forward in time. Such codes will eventually provide the needed answers about the strong field interaction and merger of the binary objects, but many technical challenges of such a computation slow the development of the needed codes. This has led us to propose as a near term alternative, the periodic standing wave (PSW) approach. Elements of this approximation have been introduced elsewhere [3, 4], but are most thoroughly presented in a recent paper[5] that we will hereafter refer to as "Paper I." In the PSW approach, a numerical solution is sought to Einstein's equations, not for a spacetime geometry evolved from initial data, but rather for sources and fields that rotate rigidly (i.e., with a helical Killing vector) and that are coupled to standing waves.

Paper I gives the details of how to extract from this solution an approximation to the problem of interest: a slowly inspiralling pair of objects coupled to outgoing waves. Paper I also describes the nature of the mathematical problem that must be solved numerically: a boundary-value problem with "standing wave boundary conditions" on a large sphere surrounding the sources. In Paper I it is pointed out that there is no *a priori* meaning to "standing wave boundary conditions" in a nonlinear problem, but that our method requires our standing wave fields to be analogous to the half-outgoing plus half-ingoing solution to a linearized problem. Two methods are given for accomplishing this in a nonlinear problem. One is the iterated Green function method whose results are presented in Paper I. The second method is to define standing waves in the weak-field radiation zone to have equal amplitude of ingoing and of outgoing waves in each multipole, and to have the relative phase of ingoing and outgoing waves adjusted so that the amplitude of the waves is the minimum compatible with the source.

A numerical implementation of this second method requires working with multipoles of the radiation field, and suggests the possibility of using a spectral method for the problem based on multipoles. Some features of the problem argue against this, however. In particular, the sources of the field are compact objects whose size is an order of magnitude less than their separation, therefore requiring a large number of terms in a multipole analysis. Worse, the coupling of the source to the field (at least in the black hole case) involves boundary conditions, or matching, on the surface of the compact objects, which would act as an inner boundary for the outer problem. Multipoles, based on coordinates with the topology of spherical polar coordinates are ill-suited to the description of such a surface.

The standard approach to this difficulty is to use coordinate patches and interpolation. In our problem such interpolation may be more difficult to implement than in other boundary value problems. In most problems involving boundary values for an elliptical problem, relaxation can be used, and interpolation can be done between steps of the relaxation iterations. Our system of partial differential equations, however, is "mixed"; it has both elliptic and hyperbolic regions inside the outer boundary. A consequence is that standard relaxation methods cannot be used. In the most straightforward numerical implementation of interpolation, therefore, the interpolation steps would have to constitute a subset of the set of equations making up the boundary value problem, and would greatly complicate the formulation and numerical solution of the problem.

In this paper we report on an alternative approach, one that has the disadvantage of adding some analytic complexity to the problem, and some worrisome features. But it is a method that gives both remarkably efficient results for model problems, and a potentially useful new approach to the coupling of moving sources to their radiation field. This new method is based on a coordinate system that is adapted to the local structure of the sources and to the large-scale structure of the distant waves. Though the standing-wave boundary condition was the original motivation for introducing an adapted coordinate system, the success with this system suggests that its utility may be more broadly applicable.

This will not be the first application of such coordinates, even in numerical relativity. "Čadež coordinates"[6], a carefully adapted coordinate system of this type, was used in much of the work on head-on collisions of black holes. Like the Čadež coordinates, our coordinate systems will reduce to source-centered spherical polar coordinates in the vicinity of the sources, and to rotation-centered spherical polar coordinates far from the sources.

The core of the usefulness of the adapted coordinates is that the field near the sources is well described by a few multipoles in these coordinates, primarily the monopole of the sources, and that the field far from the sources is well described only by a few multipoles in these coordinates. A spectral method (that is, a multipole decomposition), therefore, requires only a small number of multipoles. We will demonstrate, in fact, for mildly relativistic sources (source velocity = 30% c), excellent results are found when we keep only monopole and quadrupole terms.

There is, of course, a price to be paid for this. For one thing, there is additional analytic complexity in the set of equations. Another difficulty is the unavoidable coordinate singularity that is a feature of coordinates adapted to the two different limiting regions. Still, the potential usefulness of the method, and its early successes, have led us to use it not only for the spectral method that is the focus of this paper, but also as the coordinate system for a finite difference method, using the iterated Green function technique reported in Paper I.

B. Nonlinear model probelm

The innovative features of this new method present enough new uncertainties that it is important to study this method in the context of the simplest problem possible. We use, therefore, the same model problem as in Paper I, a simple scalar field theory with an adjustable nonlinearity. We will find it quite useful to set the nonlinearity to zero for comparison with the known solution of the linear problem, since many features of our method are unusual even for a linear problem.

For the description of our model problem, we start with Euclidean space coordinatized by the usual spherical coordinates r, θ, ϕ, and we consider sources concentrated near the points $r = a$, $\theta = \pi/2$, in the equatorial plane, and moving symmetrically according to $\phi = \Omega t$ and $\phi = \Omega t + \pi$. As in Paper I, we seek a solution of the flat-spacetime scalar field equation

$$\Psi_{;\alpha;\beta} g^{\alpha\beta} + \lambda F = \nabla^2 \Psi - \partial_t^2 \Psi + \lambda F = \text{Source}\,, \tag{1}$$

where F depends nonlinearly on Ψ, in a manner yet to be specified.

We are looking for solutions to Eq. (1) with the same helical symmetry as that of the source motions, that is, solutions for which the Lie derivative $\mathcal{L}_\xi \Psi$ is zero for the Killing vector $\xi = \partial_t + \Omega \partial_\phi$. It is useful to introduce the auxiliary coordinate $\varphi \equiv \phi - \Omega t$. In terms of spacetime coordinates t, r, θ, φ the Killing vector is simply ∂_t and the symmetry condition becomes the requirement that the scalar field Ψ is a function only of the variables r, θ and φ. (We are assuming, of course, that the form of the nonlinear term is compatible with the helical symmetry.) It is useful to consider the symmetry to be equivalent to the rule

$$\partial_t \to \; -\Omega \partial_\varphi \tag{2}$$

for scalar functions. In terms of the r, θ, φ variables, Eq. (1) for $\Psi(r, \theta, \varphi)$ takes the explicit form

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Psi}{\partial r} \right) + + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \, \frac{\partial \Psi}{\partial \theta} \right) + \left(\frac{1}{r^2 \sin^2\theta} - \Omega^2 \right) \frac{\partial^2 \Psi}{\partial \varphi^2} + \lambda F(\Psi, r, \varphi) = \text{Source}\,, \tag{3}$$

that was used in Paper I.

C. Outline

The paper is organized as follows: In Sec. II we introduce the concept of adapted coordinates and some of the details of the specific adapted coordinate system that we will use in this paper. In this section also, we present the specific form, in these coordinates, of our model nonlinear scalar equation, with ome of the explicit details relegated to the Appendix. Numerical results are given in Sec. III and onclusions are given in Sec. IV. Throughout this paper we follow the notation of Paper I[5].

II. ADAPTED COORDINATES

A. General adapted coordinates

For the definition of the adapted coordinates it is useful to introduce several Cartesian coordinate systems. We shall use the notation x, y, z to denote inertial Cartesian systems related to r, θ, ϕ in the usual way (e.g., z is the rotation axis, one of the source points moves as $x = a\cos(\Omega t)$, $y = a\sin(\Omega t)$, and so forth). We now introduce a comoving Cartesian system $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ that is related to r, θ, φ in the same way that x, y, z is related to r, θ, ϕ, that is, according to

$$\widetilde{z} = r\cos\theta \qquad \widetilde{x} = r\sin\theta\cos(\phi - \Omega t) \qquad \widetilde{y} = r\sin\theta\sin(\phi - \Omega t)\,. \tag{4}$$

In this system, as in the inertial x, y, z system, the $\widetilde{z}$ axis is the azimuthal axis. We next define the comoving system

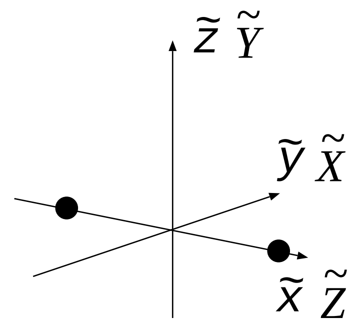

FIG. 1: Two sets of comoving Cartesian coordinates.

$$\widetilde{X} = \widetilde{y} \qquad \widetilde{Y} = \widetilde{z} \qquad \widetilde{Z} = \widetilde{x}\,, \tag{5}$$

in which the azimuthal $\widetilde{Z}$ axis is not the rotation axis, but rather is the line through the source points, as shown in Fig. 1.

Our goal now is to introduce new comoving coordinates $\chi(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$, $\Theta(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$, $\Phi(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$ that are better suited to a description of the physical problem, and that allow for more efficient computation. We will assume that the coordinate transformation is invertible, except at a finite number of discrete points, so that we may write $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$, or $\widetilde{x}, \widetilde{y}, \widetilde{z}$ as functions of χ, Θ, Φ.

In terms of the comoving Cartesian coordinates, the helical symmetry rule in Eq. (2) takes the form

$$\partial_t \to -\Omega\left(\widetilde{x}\frac{\partial}{\partial\widetilde{y}} - \widetilde{y}\frac{\partial}{\partial\widetilde{x}}\right) = -\Omega\left(\widetilde{Z}\frac{\partial}{\partial\widetilde{X}} - \widetilde{X}\frac{\partial}{\partial\widetilde{Z}}\right)\ . \tag{6}$$

Our nonlinear scalar field equation of Eq. (1) can then be written, for helical symmetry, as

$$\mathcal{L}\Psi + \lambda F = \frac{\partial^2\Psi}{\partial\widetilde{X}^2} + \frac{\partial^2\Psi}{\partial\widetilde{Y}^2} + \frac{\partial^2\Psi}{\partial\widetilde{Z}^2} - \Omega^2\left(\widetilde{Z}\frac{\partial}{\partial\widetilde{X}} - \widetilde{X}\frac{\partial}{\partial\widetilde{Z}}\right)^2\Psi + \lambda F = \text{Source}\ . \tag{7}$$

This field equation can be expressed completely in terms of adapted coordinates in the form

$$\mathcal{L}\Psi + \lambda F = A_{\chi\chi}\,\frac{\partial^2\Psi}{\partial\chi^2} + A_{\Theta\Theta}\,\frac{\partial^2\Psi}{\partial\Theta^2} + A_{\Phi\Phi}\,\frac{\partial^2\Psi}{\partial\Phi^2} + 2A_{\chi\Theta}\,\frac{\partial^2\Psi}{\partial\chi\partial\Theta} + 2A_{\chi\Phi}\,\frac{\partial^2\Psi}{\partial\chi\partial\Phi} + 2A_{\Theta\Phi}\,\frac{\partial^2\Psi}{\partial\Theta\partial\Phi}$$

$$+B_\chi\,\frac{\partial\Psi}{\partial\chi} + B_\Theta\,\frac{\partial\Psi}{\partial\Theta} + B_\Phi\,\frac{\partial\Psi}{\partial\Phi} + \lambda F = \text{Sources}\ . \tag{8}$$

It is straightforward to show that the A_{ij} and B_i coefficients here are given by

$$A_{\chi\chi} = \vec{\nabla}\chi\cdot\vec{\nabla}\chi - \Omega^2\bar{A}_{\chi\chi} \tag{9}$$
$$A_{\Theta\Theta} = \vec{\nabla}\Theta\cdot\vec{\nabla}\Theta - \Omega^2\bar{A}_{\Theta\Theta} \tag{10}$$
$$A_{\Phi\Phi} = \vec{\nabla}\Phi\cdot\vec{\nabla}\Phi - \Omega^2\bar{A}_{\Phi\Phi} \tag{11}$$
$$A_{\chi\Theta} = \vec{\nabla}\chi\cdot\vec{\nabla}\Theta - \Omega^2\bar{A}_{\chi\Theta} \tag{12}$$
$$A_{\chi\Phi} = \vec{\nabla}\chi\cdot\vec{\nabla}\Phi - \Omega^2\bar{A}_{\chi\Phi} \tag{13}$$
$$A_{\Theta\Phi} = \vec{\nabla}\Theta\cdot\vec{\nabla}\Phi - \Omega^2\bar{A}_{\Theta\Phi} \tag{14}$$
$$B_{\chi} = \vec{\nabla}^2\chi - \Omega^2\bar{B}_{\chi} \tag{15}$$
$$B_{\Theta} = \vec{\nabla}^2\Theta - \Omega^2\bar{B}_{\Theta} \tag{16}$$
$$B_{\Phi} = \vec{\nabla}^2\Phi - \Omega^2\bar{B}_{\Phi}\ . \tag{17}$$

Here the gradients, Laplacians and dot products are to be taken treating the $\widetilde{X}$,$\widetilde{Y}$,$\widetilde{Z}$ as Cartesian coordinates, so that, for example,

$$\vec{\nabla}\chi\cdot\vec{\nabla}\Theta = \frac{\partial\chi}{\partial\widetilde{X}}\frac{\partial\Theta}{\partial\widetilde{X}} + \frac{\partial\chi}{\partial\widetilde{Y}}\frac{\partial\Theta}{\partial\widetilde{Y}} + \frac{\partial\chi}{\partial\widetilde{Z}}\frac{\partial\Theta}{\partial\widetilde{Z}}\,. \tag{18}$$

The form of the $\bar{A}_{ij}$ and $\bar{B}_j$ terms in Eqs. (9)–(17) are given, for general adapted coordinates, in Eqs. (A17) – (A27).

B. A specific adapted coordinate system: TCBC

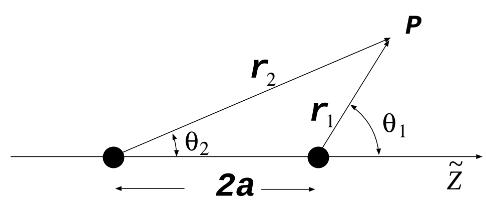

FIG. 2: Adapted coordinates of a point P in two spatial dimensions

Before discussing general features of an adapted coordinate system, it will be useful to give a specific example. For that example, we choose a coordinate system χ,Θ,Φ that is particularly simple in form, though (as we shall discuss below) not the choice that is numerically most efficient. The chosen coordinates are most easily understood by starting with the distances r_1 and r_2 from the source points, and with the angles θ_1, θ_2 shown in Fig. 2. The formal definitions of the adapted coordinates are

$$\chi \equiv \sqrt{r_1 r_2} = \left\{\left[\left(\widetilde{Z}-a\right)^2+\widetilde{X}^2+\widetilde{Y}^2\right]\left[\left(\widetilde{Z}+a\right)^2+\widetilde{X}^2+\widetilde{Y}^2\right]\right\}^{1/4} \tag{19}$$
$$\Theta \equiv \frac{1}{2}\left(\theta_1+\theta_2\right) = \frac{1}{2}\tan^{-1}\left(\frac{2\widetilde{Z}\sqrt{\widetilde{X}\widetilde{Y}}}{\widetilde{Z}^2-a^2-\widetilde{X}^2-\widetilde{Y}^2}\right) \tag{20}$$
$$\Phi \equiv \tan^{-1}\left(\widetilde{X}/\widetilde{Y}\right)\,. \tag{21}$$

This choice is sometimes called “two-center bipolar coordinates”[7], hereafter TCBC, and is equivalent to the zero-order coordinates used by Čadež[6].

An attractive feature of this particular choice of adapted coordinates is that the above relationships can be inverted in simple closed form to give

$$\widetilde{Z} = \sqrt{\frac{1}{2}\left[a^2+\chi^2\cos 2\Theta+\sqrt{\left(a^4+2a^2\chi^2\cos 2\Theta+\chi^4\right)}\,\right]} \tag{22}$$

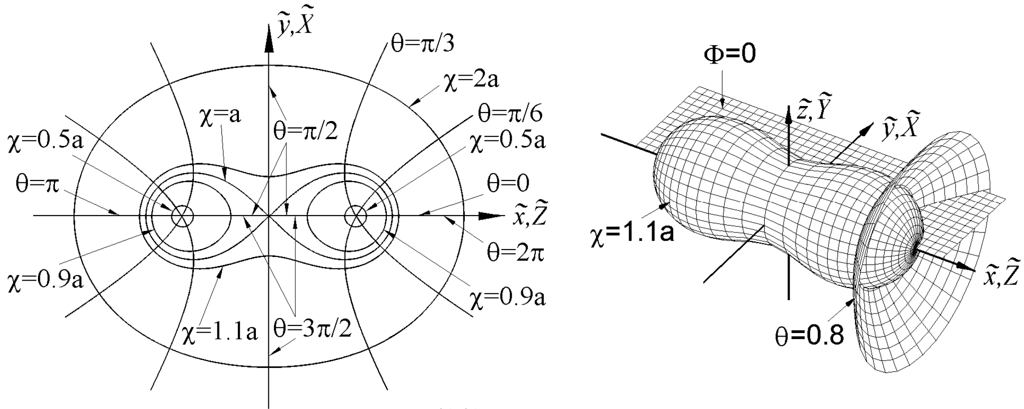

FIG. 3: Adapted coordinates in the $\widetilde{x},\widetilde{y}$ plane, and three-dimensional coordinate surfaces.

$$\widetilde{X} = \sqrt{\frac{1}{2}\left[-a^2 - \chi^2\cos 2\Theta + \sqrt{(a^4 + 2a^2\chi^2\cos 2\Theta + \chi^4)}\right]}\ \cos\Phi\ . \tag{23}$$

$$\widetilde{Y} = \sqrt{\frac{1}{2}\left[-a^2 - \chi^2\cos 2\Theta + \sqrt{(a^4 + 2a^2\chi^2\cos 2\Theta + \chi^4)}\right]}\ \sin\Phi\ . \tag{24}$$

The meaning of the χ, Θ coordinates in the $\widetilde{x},\widetilde{y}$ plane (the $\widetilde{Z},\widetilde{X}$ plane) is shown on the left in Fig. 3; a picture of three-dimensional χ, Θ, and Φ surfaces is shown on the right.

The geometrical definition inherent in Fig. 2 suggests that the adapted coordinate surfaces have the correct limit far from the sources. This is confirmed by the limiting forms Eqs. (22)–(24) for $\chi \gg a$. Aside from fractional corrections of order a^2/χ^2 the relations are

$$\widetilde{Z} \to \chi\cos\Theta \qquad \widetilde{X} \to \chi\sin\Theta\cos\Phi \qquad \widetilde{Y} \to \chi\sin\Theta\sin\Phi\ . \tag{25}$$

Near the source point at $\widetilde{Z} = \pm a$, the limiting forms, aside from fractional corrections of order χ^2/a^2, are

$$\widetilde{Z} \to \pm a + \frac{\chi^2}{2a}\cos(2\Theta) \qquad \widetilde{X} \to \frac{\chi^2}{2a}\sin(2\Theta)\cos\Phi \qquad \widetilde{Y} \to \frac{\chi^2}{2a}\sin(2\Theta)\sin\Phi\,. \tag{26}$$

These limits, and Fig. 2 show that near the source point at $\widetilde{Z} = a$ the expression $\chi^2/2a$ plays the role of radial distance, and 2Θ plays the role of polar coordinate. (Near the source point at $\widetilde{Z} = -a$, the expression $\chi^2/2a$ again plays the role of radius, but the polar angle is $\pi - 2\Theta$.) Notice that both for the near and the far limit, the polar angle is defined with respect to the line through the sources, the $\widetilde{Z}$ axis, not with respect to the rotational $\widetilde{z}$ axis.

It is clear that our new system has a coordinate singularity at the origin. Indeed, there must be a coordinate singularity in any such adapted coordinate system. The switch from the small-χ coordinate surfaces, disjoint 2-spheres around the sources, to the large-χ single 2-sphere cannot avoid a singularity.

The remaining specification needed is the outer bounary conditions on some large approximate spherical surface $\chi = \chi_{\rm max}$. The usual Sommerfeld outgoing outer boundary condition $\partial_t\psi = -\partial_r\psi$, can be approximated as $\partial_t\psi = -\partial_\chi\psi$. The fractional error introduced by this substitution is of order a^2/χ^2. The Sommerfeld condition itself is accurate only up to order wavelength/r. Since the wavelength is larger than a, our substitution $r \to \chi$ in the outer boundary condition introduces negligibly small errors. To apply the helical symmetry we use the replacement rule in Eq. (6) and the outgoing boundary condition becomes

$$\frac{\partial\Psi}{\partial\chi} = \Omega\left(\widetilde{Z}\frac{\partial\Psi}{\partial\widetilde{X}} - \widetilde{X}\frac{\partial\Psi}{\partial\widetilde{Z}}\right) = \Omega\left(\Gamma^\Theta\frac{\partial\Psi}{\partial\Theta} + \Gamma^\Phi\frac{\partial\Psi}{\partial\Phi} + \Gamma^\chi\frac{\partial\Psi}{\partial\chi}\right). \tag{27}$$

where the Γs are given explicitly in Appendix A. At large χ the outgoing condition can be written

$$\frac{\partial\Psi}{\partial\chi} = \Omega\left(\cos\Phi\frac{\partial\Psi}{\partial\Theta} - \frac{\cos\Theta}{\sin\Theta}\sin\Phi\frac{\partial\Psi}{\partial\Phi}\right)\left(1 + \mathcal{O}(a^2/\chi^2)\right)\ . \tag{28}$$

The correction on the right is higher-order at the outer boundary $\chi = \chi_{\rm max}$ and can be ignored. The ingoing boundary condition follows by changing the sign of the right hand side of Eq. (27) or (28).

The problem of Eqs. (8) and (28) is a well-posed boundary-value problem analagous to that in Paper I[5]. As in Paper I, this problem can be numerically implemented using the finite difference method (FDM) of discretizing derivatives. The difference between such a computation and that of Paper I is, in principle, only in the coordinate dependence of the coefficients ($A_{\chi\chi}$, $A_{\Theta\Theta}, \cdots$) in the differential equation and the outer boundary condition.

C. Requirements for adapted coordinates

For the scalar problem, there are obvious advantage of the coordinate system pictured in Fig. 3. First, the surfaces of constant χ approximate the surfaces of constant Ψ near the sources, where field gradients are largest, and where numerical difficulties are therefore expected. Since the variation with respect to Θ and Φ is small on these surfaces, finite differencing of Θ and Φ derivatives should have small truncation error. The steep gradients in χ, furthermore, can be dealt with in principle by a reparameterization of χ to pack more grid points near the source points. An additional, independent advantage to the way the coordinates are adapted to the source region is that these coordinates are well suited for the specification of inner boundary conditions on a constant χ surface. Because of these advantages we shall reserve the term "adapted" to a coordinate system for which constant χ surfaces near the source approximate spheres concentric with the source.

A second feature of the TCBC coordinates that we shall also require in general, is that in the region far from the sources, χ,Θ,Φ asymptotically approach spherical coordinates, the coordinates best suited for describing the radiation field. If the approach to spherical coordinates is second-order in r/a, then the outgoing boundary conditions will be that in Eq. (28).

There are practical considerations that also apply to the choice of adapted coordinates. The coefficients of the rotational terms in the equation (i.e., those involving $\bar{A}_{ij}$ and $\bar{B}_i$) require computing second derivatives of the transformation from Cartesian to adapted coordinates. If those relationships are only known numerically, these second derivatives will tend to be noisy. For that reason, a desirable and perhaps necessary feature of the adapted coordinates is therefore that closed form expressions exist for $\chi(\widetilde{x}, \widetilde{y}, \widetilde{z})$, and $\Theta(\widetilde{x}, \widetilde{y}, \widetilde{z})$. (The expression for Φ, the azimuthal angle about the line through the source points, is trivial.) It is possible in principle, of course, to have the adapted coordinates defined without respect to the Cartesians. In the scalar model problem, the coordinates could be defined by giving the form of the flat spacetime metric in these coordinates. The nature of the helical Killing symmetry, analogous to Eq. (6) would still have to be specified of course. The choice of adapted coordinates becomes a much richer subject in the case of the gauge-fixed general relativity problem that is the ultimate goal of the work, for which see **Chris???**

The TCBC coordinates satisfy all the practical requirements of an adapted coordinate system. In particular, the functions $\chi(\widetilde{x}, \widetilde{y}, \widetilde{z})$, and $\Theta(\widetilde{x}, \widetilde{y}, \widetilde{z})$, as well as the inverses are all explicitly known in terms of elementary functions. Though the TCBC coordinates are therefore convenient, in addtition to being well suited to the problem in Eq. (1), they are not optimal. The perfect coordinates would seem to be those for which the constant χ surfaces agree exactly with the constant Ψ surfaces. Since of course is impossible in practice (and, in addition, would not be compatible with the requirement that the coordinates go asymptotically to spherical coordinates). We should therefore modify the criterion for the "perfect coordinates" to be those for which Ψ is constant on constant χ surfaces for no rotation ($\Omega = 0$). The TCBC coordinate system, in fact, does satisfy that requirement for the version of the problem in Eq. (1) in two spatial dimensions with no nonlinearity. (The logarithm of χ is a harmonic function of $\widetilde{x}$, $\widetilde{y}$.) Due to this "near perfection" of the TCBC coordinates for the linear two-dimensional' problem we found found that we were able to achieve very good accuracy for that case with moderate degrees of rotation.

This suggests that we could achieve an improvement over the TCBC coordinates, by choosing χ to be proportional to solutions of the nonrotating case of Eq. (1) in three spatial dimensions. Since the nonlinear case would result in a solution that is known only numerically, we can follow the pattern of the two-dimensional case and choose χ simply to be proportional to the solution of the linear nonrotating three-dimensional problem. The Θ coordinate that is orthogonal to this χ would have to be found numerically, and would therefore be troublesome. But there is no need for Θ and χ to be orthogonal. We could, therefore, use the TCBC definition of Θ in Eq. (20). An improved set of adapted coordinates, then, would seem to be

$$\chi = \frac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \qquad \Theta = \frac{1}{2}\left(\theta_1 + \theta_2\right) , \tag{29}$$

where r_i, θ_i are the distances and angles shown Fig. 2.

In this paper, we shall report only numerical results from the simplest adapted coordinate system to implement, the TCBC coordinates. There are two reasons for this. The first is the obvious advantages of working with the simplicity of the TCBC case, and the advantage of having simple explicit expressions for all coefficients in Eq. (8). The second reason that we do not use the apparently superior adapted coordinate in Eq. (29), is that we do not expect there to

TABLE I: The reduction of the wave amplitude due to the nonlinearity. A simple estimate is compared with FDM computations in spherical and in TCBC coordinates.

λ	Estimate	Spherical	TCBC
-1	69%	78%	??%
-2	62%	68%	??%
-5	53%	55%	??%
-10	46%	47%	??%
-25	37%	35%	??%

be an equivalent for the general relativity problem. In that case there will be several different unknown fields to solve for, and there is no reason to think that the optimal coordinate system for one of the fields will be the same for the others.

D. Nonlinear scalar model in adapted coordinates

We use the same nonlinear term that we did in Paper I

$$\lambda F = \lambda \frac{\Psi^5}{\Psi_0^4 + \Psi^4} . \tag{30}$$

The parameter λ allows us to adjust the imortance of the nonlinearity, and to turn it off to explore linear solutions. The

To identify a physical problem, we must specify the source term, or its equivalent, in Eq. (8). In Paper I we used point-like source terms. With our adapted coordinates we can formulate the problem in a more physical way: We set the source to zero, and we supply inner boundary conditions at the approximate spherical surface for some $\chi = \chi_0$, where χ/a is small.

We have carried out such a computation, but with an additional difference from the computation in Paper I. Rather than include an explicit, inhomogeneous source term in the field equation (8), we have used Dirichlet data on a constant-χ surface $\chi = \chi_0 \ll a$. This data has been chosen to be that for the linearized solution from two point sources. That solution is expressible in closed form as a series involving spherical harmonics and spherical Bessel functions of r, θ, φ coordinates[5]. Since these coordinates can be found from χ, Θ, Φ with Eqs. (4), (5) and (22)– (24), we can compute the "exact" linearized Ψ at $\chi = \chi_0$, and use this as inner boundary data. This approach is more appropriate for the eventual application to black hole sources; in Paper I an explicit source was used because inner boundary conditions would have been difficult to impose in spherical coordinates.

Aside from the differences already mentioned, our FDM computatioins repeated the numerical techniques (direct matrix inversion, Newton-Raphson iteration, continuation) of Paper I, and are not presented here. Two points, however, are worth mentioning. First, there were no fundamental difficulties in computing solutions; the only difficulties were essentially the same difficulties encountered with the FDM using spherical coordinates in Paper I. This expected result confirmed the fact that there is nothing ill-posed in the boundary value problem in the adapted coordinates.

The second point worth mentioning concerns a feature of the problem that may indeed seem ill-posed: the coordinate singularity at $\chi = a$, $\Theta = \Pi/2$ (see Fig. 3). We have found that the truncation error at these points dominates the computational error, and causes the code to be only first-order convergent. A study has been completed of how to remove this dominant error, and restore second-order convergence, in a relatively simple way, by introducing finite elements in the region of the singularity[8].

Include a discussion of the Yukawa-¿Coulomb nature of the fields@.

III. NUMERICAL RESULTS

Table II is equivalent to the analogous table in Paper I. It gives information on the numerical errors of the most computationally intensive solution type: that for strongly nonlinear waves computed via Newton-Raphson iteration. A single physical model ($\lambda = -10$, $\Psi_0 = 0.15$, $a\Omega = 0.3$, outer boundary at $r = 30a$) is computed on five different grids. As a measure of the truncation error for grid k, the L2 difference (the square root of the average square difference) is found between the results for grid k and for grid $k+1$. This is listed in Table II as the error in grid k. These results, especially for the finest three grids, suggest quadratic convergence of the numerical process.

TABLE II: Convergence of finite difference computations in TCBC. The true results have not yet been filled in.. these results are from Paper I

k	$n_\chi \times n_\Theta \times n_\Phi$	Error
1	90×10×16	2.71 E-5
2	120×14×22	1.60 E–5
3	150×16×26	8.68 E-6
4	180×20×32	5.22 E-6
5	210×24×38	

FIG. 4: Extracted outgoing nonlinear waves vs. true outgoing nonlinear waves.

IV. CONCLUSIONS

Thoughts: conclusions should be pretty much limited to adapted coordinates.

- The adapted coordinates lead to a well-posed boundary value problem. These coordinates are well suited to the specification of inner boundary conditions rather than an explicit source term. The boundary value problem can be solved by FDM as in Paper I.
- There may be better adapted coordinates. Details about 2D linear and solution of harmonic, but in GR......
- To come: spectral method.

APPENDIX A: COEFFICIENTS FOR ADAPTED COORDINATES

As in the two-dimensional case, the dot products $\vec{\nabla}\chi\cdot\vec{\nabla}\Theta$, $\vec{\nabla}\chi\cdot\vec{\nabla}\Phi$, and $\vec{\nabla}\Theta\cdot\vec{\nabla}\Phi$, vanish since the adapted coordinates are orthorgonal (with respect to a Cartesian metric on $\widetilde{X}$,$\widetilde{Y}$,$\widetilde{Z}$). The other dot products and Laplacians are evaluated with the explicit transformations in Eqs. (19)–(24), from which we find

$$\nabla^2\chi = \frac{a^2+2Q}{\chi^3} \tag{A1}$$

$$\nabla^2\Theta = \frac{\sqrt{Q+a^2+\chi^2\cos(2\,\Theta)}}{\sqrt{Q-a^2-\chi^2\cos(2\,\Theta)}}\,\frac{\left(Q-a^2\right)}{\chi^4} \tag{A2}$$

$$\nabla^2\Phi = 0 \tag{A3}$$

$$\vec{\nabla}\chi\cdot\vec{\nabla}\chi = \frac{Q}{\chi^2} \tag{A4}$$

$$\vec{\nabla}\Theta\cdot\vec{\nabla}\Theta = \frac{Q}{\chi^4} \tag{A5}$$

$$\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi = 2\,\frac{Q+a^2+\chi^2\cos(2\,\Theta)}{\chi^4\sin^2(2\,\Theta)} \tag{A6}$$

FIG. 5: Extracted outgoing nonlinear waves vs. true outgoing nonlinear waves at small values of χ

where Q is the function

$$Q \equiv \sqrt{a^4 + 2\, a^2 \chi^2 \cos(2\Theta) + \chi^4}\ . \tag{A7}$$

In general the $\bar{A}$ and $\bar{B}$ terms are computed from the following:

$$\bar{A}_{\chi\chi} = Z^2 \left(\frac{\partial \chi}{\partial X}\right)^2 + X^2 \left(\frac{\partial \chi}{\partial Z}\right)^2 - 2XZ \left(\frac{\partial \chi}{\partial X}\right)\left(\frac{\partial \chi}{\partial Z}\right) \tag{A8}$$

$$\bar{A}_{\Theta\Theta} = Z^2 \left(\frac{\partial \Theta}{\partial X}\right)^2 + X^2 \left(\frac{\partial \Theta}{\partial Z}\right)^2 - 2XZ \left(\frac{\partial \Theta}{\partial X}\right)\left(\frac{\partial \Theta}{\partial Z}\right) \tag{A9}$$

$$\bar{A}_{\Phi\Phi} = Z^2 \left(\frac{\partial \Phi}{\partial X}\right)^2 + X^2 \left(\frac{\partial \Phi}{\partial Z}\right)^2 - 2XZ \left(\frac{\partial \Phi}{\partial X}\right)\left(\frac{\partial \Phi}{\partial Z}\right) \tag{A10}$$

$$\bar{A}_{\chi\Theta} = Z^2 \left(\frac{\partial \chi}{\partial X}\right)\left(\frac{\partial \Theta}{\partial X}\right) + X^2 \left(\frac{\partial \chi}{\partial Z}\right)\left(\frac{\partial \Theta}{\partial Z}\right) - XZ \left[\left(\frac{\partial \chi}{\partial Z}\right)\left(\frac{\partial \Theta}{\partial X}\right) + \left(\frac{\partial \chi}{\partial X}\right)\left(\frac{\partial \Theta}{\partial Z}\right)\right] \tag{A11}$$

$$\bar{A}_{\chi\Phi} = Z^2 \left(\frac{\partial \chi}{\partial X}\right)\left(\frac{\partial \Phi}{\partial X}\right) + X^2 \left(\frac{\partial \chi}{\partial Z}\right)\left(\frac{\partial \Phi}{\partial Z}\right) - XZ \left[\left(\frac{\partial \chi}{\partial Z}\right)\left(\frac{\partial \Phi}{\partial X}\right) + \left(\frac{\partial \chi}{\partial X}\right)\left(\frac{\partial \Phi}{\partial Z}\right)\right] \tag{A12}$$

$$\bar{A}_{\Theta\Phi} = Z^2 \left(\frac{\partial \Theta}{\partial X}\right)\left(\frac{\partial \Phi}{\partial X}\right) + X^2 \left(\frac{\partial \Theta}{\partial Z}\right)\left(\frac{\partial \Phi}{\partial Z}\right) - XZ \left[\left(\frac{\partial \Theta}{\partial Z}\right)\left(\frac{\partial \Phi}{\partial X}\right) + \left(\frac{\partial \Theta}{\partial X}\right)\left(\frac{\partial \Phi}{\partial Z}\right)\right] \tag{A13}$$

$$\bar{B}_{\chi} = Z^2 \left(\frac{\partial^2 \chi}{\partial X^2}\right) + X^2 \left(\frac{\partial^2 \chi}{\partial Z^2}\right) - 2XZ \left(\frac{\partial^2 \chi}{\partial X \partial Z}\right) - X \left(\frac{\partial \chi}{\partial X}\right) - Z \left(\frac{\partial \chi}{\partial Z}\right) \tag{A14}$$

$$\bar{B}_{\Theta} = Z^2 \left(\frac{\partial^2 \Theta}{\partial X^2}\right) + X^2 \left(\frac{\partial^2 \Theta}{\partial Z^2}\right) - 2XZ \left(\frac{\partial^2 \Theta}{\partial X \partial Z}\right) - X \left(\frac{\partial \Theta}{\partial X}\right) - Z \left(\frac{\partial \Theta}{\partial Z}\right) \tag{A15}$$

$$\bar{B}_{\Phi} = Z^2 \left(\frac{\partial^2 \Phi}{\partial X^2}\right) + X^2 \left(\frac{\partial^2 \Phi}{\partial Z^2}\right) - 2XZ \left(\frac{\partial^2 \Phi}{\partial X \partial Z}\right) - X \left(\frac{\partial \Phi}{\partial X}\right) - Z \left(\frac{\partial \Phi}{\partial Z}\right) \tag{A16}$$

In the case of the TCBC coordinates defined in Eqs. (19) – (24), the explicit forms of the coefficients are

$$\bar{A}_{\chi\chi} = \frac{a^4 \sin^2(2\Theta)\ \cos^2\Phi}{\chi^2} \tag{A17}$$

$$\bar{A}_{\Theta\Theta} = \frac{\cos^2\Phi\ \left[\chi^2 + a^2\cos(2\Theta)\right]^2}{\chi^4} \tag{A18}$$

$$\bar{A}_{\Phi\Phi} = \sin^2\Phi\ \frac{Q + a^2 + \chi^2\cos(2\Theta)}{Q - a^2 - \chi^2\cos(2\Theta)} \tag{A19}$$

$$\bar{A}_{\chi\Theta} = \frac{a^2\ \left[\chi^2 + a^2\cos(2\Theta)\right]\ \sin(2\Theta)\cos^2\Phi}{\chi^3} \tag{A20}$$

$$\bar{A}_{\chi\Phi} = -\frac{a^2\ \left[Q + a^2 + \chi^2\cos(2\Theta)\right]\ \sin\Phi\cos\Phi}{\chi^3} \tag{A21}$$

$$\bar{A}_{\Theta\Phi} = -\frac{\sin(\Phi)\cos(\Phi)\left[a^2 + \chi^2\cos(2\,\Theta) + Q\right]\left[\chi^2 + a^2\cos(2\,\Theta)\right]}{\chi^4\sin(2\,\Theta)} \tag{A22}$$

$$\bar{B}_\chi = \frac{a^2\left[\cos^2(\Phi)\left\{3a^2\cos^2(2\Theta) - Q - 2a^2 + \chi^2\cos(2\Theta)\right\} + Q + a^2 + \chi^2\cos(2\Theta)\right]}{\chi^3} \tag{A23}$$

$$\bar{B}_\Phi = \frac{\left(3Q + a^2 + \chi^2\cos 2\Theta\right)\sin(\Phi)\cos(\Phi)}{Q - a^2 - \chi^2\cos 2\Theta} \tag{A24}$$

$$\bar{B}_\Theta = \frac{\sqrt{Q + a^2 + \chi^2\cos(2\Theta)}}{\chi^6\sqrt{Q - a^2 - \chi^2\cos(2\Theta)}}\ (c\cos^2\Phi + d) \tag{A25}$$

where

$$c \equiv a^2\chi^4\cos(2\,\Theta) + 2\,a^4\chi^2 + 4\,a^6\cos(2\,\Theta) + 4\,a^4\chi^2\left(\cos(2\,\Theta)\right)^2 - 4\,a^4 Q\cos(2\,\Theta) - 2\,a^2 Q\chi^2 - \chi^6 \tag{A26}$$

$$d \equiv \chi^4\left(a^2\cos(2\,\Theta) + \chi^2\right) \tag{A27}$$

The coefficients needed in the Sommerfeld boundary condition Eq. (27) are

$$\Gamma^\Theta = \frac{\chi^2 + a^2\cos 2\Theta}{\chi^2}\ \cos\Phi = \cos\Phi\left(1 + \mathcal{O}(a^2/\chi^2)\right) \tag{A28}$$

$$\Gamma^\Phi = -\sqrt{\frac{Q + a^2 + \chi^2\cos(2\Theta)}{Q - a^2 - \chi^2\cos(2\Theta)}}\ \sin\Phi = -\cot\Theta\sin\Phi\left(1 + \mathcal{O}(a^2/\chi^2)\right) \tag{A29}$$

$$\Gamma^\chi = \frac{1}{\chi^3}\ \sqrt{\left[Q + a^2 + \chi^2\cos(2\Theta)\right]\left[Q - a^2 - \chi^2\cos(2\Theta)\right]} = \frac{2\sin 2\Theta}{\chi}\left(1 + \mathcal{O}(a^2/\chi^2)\right)\ . \tag{A30}$$

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